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Optimal and Near-Optimal Mechanism Design with Interdependent Values

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We study optimal and approximately-optimal mechanism design questions in the interdependent values model, which generalizes the standard setting of independent and private values. We focus our attention on ex post incentive compatible and individually rational mechanisms, and develop an analog of Myerson's optimal auction theory that applies to many interdependent settings of interest. We demonstrate two applications for specific interdependent settings: First, a parallel result to the well-known optimality of the second-price auction with reserve for i.i.d. bidders, where the English auction replaces the second-price one. Second, we identify good prior-independent auctions — auctions with near-optimal expected revenue across a wide range of priors — for certain interdependent value settings.

Categories and Subject Descriptors: [**Theory of Computation**]: Algorithmic game theory and mechanism design

Additional Key Words and Phrases: interdependence; optimal auctions; prior-independence

1. INTRODUCTION

The subject of this paper is optimal and approximately-optimal mechanism design, where bidders have *interdependent* values, and the mechanism has *ex post* incentive compatibility and individual rationality guarantees.

Economic research on auctions has explored different valuation models over the past decades, which can roughly be divided into private versus interdependent values (for example, see [Krishna 2010, Chapters 2-5] versus [Krishna 2010, Chapters 6-10]). The more nacent research effort in theoretical computer science has focused largely on the restricted setting of *independent private* values, recently also venturing into the realm of *correlated private* values [Papadimitriou and Pierrakos 2011; Dobzinski et al. 2011; Cai et al. 2012]. A broad research goal is therefore to apply the computer science lens to the study of mechanisms for the general interdependent setting.

For many auctions that arise in practice, interdependent values are a more realistic model of bidders' values than private values. Interdependence models situations in which every bidder has only partial information, called his *signal*, about his value for winning the auction. Furthermore, other bidders may possess information that affects the bidder's value — mathematically his value is a function of his own signal and the signals of his competitors. A classic example from the economic literature is the *mineral rights* model [Wilson 1969]. In auctions for oil drilling, the value of the drilling rights is determined by whether or not there is oil to be found on the drilling site. This value is common – yet unknown – to all bidders. However, typically every bidder has some private noisy signal regarding the value,

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Table I. Terminology

	Private	Common
Independent	Independent private values	Independent common values
Correlated	Correlated private values	Interdependent values

achieved by, for example, conducting a geological survey. The information gathered by the other bidders would certainly change a bidder's expected value for winning if he gains access to it.

In the context of optimal mechanism design with interdependent values, it is well-known that there is a crucial difference between a mechanism which is ex post incentive compatibile (IC) and individually rational (IR), and one which guarantees truthfulness and individual rationality in the Bayesian sense. Cremer and McLean [1985; 1988] have shown a Bayesian optimal mechanism that extracts full social welfare under (arguably) mild non-degeneracy conditions. Less is known regarding ex post IC and IR optimal mechanisms for interdependent values, and their study is one of our goals.

1.1. Our Results

This paper's contributions are as follows (from now on we use the terminology in Table I):

(1) We develop a general analog of Myerson's optimal auction theory that applies to many interdepedent settings of interest. While Myerson's theory does not hold in general for interdependent values (indeed, there are settings in which the Cremer-McLean mechanism extracts higher revenue than the Myerson mechanism), we show it is partially recovered when we impose ex post rather than Bayesian IC and IR constraints.

We first apply standard techniques to characterize ex post IC and IR mechanisms in the interdependent model and to show that their expected revenue equals their expected "conditional virtual surplus," thus unifying many partial characterizations that appear in the literature (see related work in Section 4). We use the characterization to identify sufficient conditions under which the simple, "ironless" form of the Myerson mechanism is optimal. Under these conditions, the optimal mechanism simply allocates to the bidders with highest non-negative (conditional) virtual values. Our cleaner result is in the correlated private values setting, where the standard conditions of regularity and affiliation of the signals are sufficient. For the general interdependent setting, more stringent conditions are identified.

(2) We apply our results above to prove an optimal auction result in the correlated private values setting — that the English (ascending) auction with a suitable reserve price is optimal among all ex post IC and IR auctions under the same, arguably mild, assumptions of affiliation and regularity, and with an additional symmetry condition. This result is parallel to the optimality of second price auction with reserve for regular and symmetric bidders in the standard (independent private) model. A similar result requiring more assumptions is known for interdepedent values by the work of Lopomo [2000].

(3) For certain interdependent value settings, we develop a *prior-independent* auction — an auction that is simultaneously near-optimal across a range of prior distributions. Such auctions are useful if good prior information is expensive or impossible to acquire, or if a single procedure is to be re-used several times, in settings with different or not-yet-known prior distributions. We adapt the single sample approach of [Dhangwatnotai et al. 2010] to our setting, but show that we need an additional MHR assumption for it to apply (and as a result, a different analysis).

Our prior-independence result demonstrates that non-trivial research questions arise even in the simplest interdependent settings. Our Myerson-like characterization suggests that many interesting mechanism design results should be possible, even when bidders have interdependent values.

1.2. Organization

In Sections 2 and 3 we present the model and solution concept. In Section 4 we discuss related work. Section 5 includes technical results (1) and (2) above. Result (3) appears in Section 6. The study of interdependent settings raises many further research directions; several open questions appear in the appendix.

2. MODEL

2.1. Interdependent Values Setting

We consider single-parameter Bayesian auction environments $(E, \mathcal{I})$, where $E = \{1, \ldots, n\}$ is a set of bidders, and $\mathcal{I} \subseteq 2^E$ is a non-empty collection of *feasible* bidder subsets, i.e., subsets of bidders who can win the auction simultaneously. $(E, \mathcal{I})$ is a downward-closed set system, in which a subset of a feasible subset is also feasible. A canonical example of a single-parameter environment is a multi-unit auction with unit-demand bidders, for which $\mathcal{I}$ is all sets of bidders for which there are enough units. Most of our results are new and interesting even for single-item auctions.

The bidders have possibly correlated, private *signals* $s_1, \ldots, s_n$, drawn from a joint distribution F with density f over the support $[0, \omega_i]^n$ (ω_i may be ∞). We adopt the standard assumptions that f is continuous and nowhere zero. Every bidder i has a *valuation function* v_i whose arguments are the signals, and his *interdependent value* for winning is $v_i(\vec{s})$. Interdependent values are also called *information externalities* among the bidders. We impose the following standard assumptions on the valuation function $v_i(\cdot)$:

— Non-negative and normalized ($v_i(\vec{0}) = 0$);
— Twice continuously differentiable;
— Non-decreasing in all variables, strictly increasing in s_i.

We also assume, as is standard, finite expectation $\mathbb{E}_{\vec{s}}[v_i(\vec{s})] < \infty$.

Notation. Fix a signal profile s_{-i}. Let $v_{i|s_{-i}}(\cdot)$ denote bidder i's value given s_{-i} as a function of his signal s_i. Observe that $v_{i|s_{-i}}(\cdot)$ is invertible; denote by $v^{-1}_{i|s_{-i}}(\nu)$ or $v_i^{-1}(\nu \mid s_{-i})$ the signal s_i such that $v_{i|s_{-i}}(s_i) = \nu$. Slightly abusing notation, given s_{-i} denote the derivative of $v_{i|s_{-i}}(\cdot)$ at s_i by

$$\frac{d}{ds_i} v_i(\vec{s}) = \frac{d}{ds_i} v_{i|s_{-i}}(s_i).$$

Other value models. The described interdependent values model is very general; it encompasses several narrower settings of interest:

(1) *Correlated private values* settings, in which $v_i(\vec{s}) = s_i$ for every i;
(2) *Independent common*[1] *values* settings, in which the signal distribution F is a product distribution of its marginals $\{F_i\}$;
(3) *Independent private values* settings, in which both $v_i(\vec{s}) = s_i$ for every i and F is a product distribution.

2.2. Motivating Examples

We describe two natural examples of common values. In the first example, bidders' values directly depend on the private preferences of the others. In the second example, bidders' values depend on a hidden stochastic "state of the world", of which others may posses private knowledge.

[1]This terminology is due to Klemperer [1999], and is not to be confused with a *pure common value* setting in which $v_i(\vec{s})$ is equal among all bidders.

Example 2.1 (*Weighted-sum values [Li 2013]*). Let $\beta \in [0,1]$. Every bidder's value is a sum of his own signal and a weighted sum of the other signals:

$$v_i(\vec{s}) = s_i + \beta \sum_{j \neq i} s_j.$$

When $\beta = 1$, this results in the *wallet game* [Klemperer 1998]. Weighted-sum values can model, e.g., values for a painting in an auction: A bidder's value for the painting is determined by his own appreciation of it, combined with the painting's "resale value" based on how much others appreciate it.

Example 2.2 (*Mineral rights model [Wilson 1969; Krishna 2002]*). Bidders have a hidden stochastic pure common value v, modeled by a random variable V drawn from a publicly known distribution F_V. For concreteness let F_V be a truncated normal distribution with known parameters μ, σ (assume μ is far from 0 and σ is small). The signals are $s_i = v + n_i$, where the n_i's are i.i.d. samples drawn from a normal distribution with $\mu_N = 0$ and some small σ_N. An important feature of the mineral rights model is that conditional on the event $V = v$, bidders' signals are independent. Also, each signal is an unbiased estimator of V (its expectation given $V = v$ is v). The bidders' effective pure common value — their value for all operational purposes — is

$$v_i(\vec{s}) = \mathbb{E}_{V \sim F_V}[V \mid \vec{s}].$$

The mineral rights model was developed to capture values in auctions for an oil drilling lease: Values are determined by the true amount of oil to be found, but uncertainty about this amount creates interdependency among them.

We now give an example of a correlated signal distribution.

Example 2.3 (*Multivariate normal distribution*). Signals are drawn from a multivariate normal distribution, whose marginal distributions are normal too. For reasons that will become clear below, assume that the distribution is symmetric and that all pairwise covariances are nonnegative.

2.3. Conditional Virtual Values and Regularity

Fix a bidder i and a signal profile s_{-i}. The *conditional marginal density* $f_i(\cdot \mid s_{-i})$ of bidder i's signal given s_{-i} is

$$f_i(s_i \mid s_{-i}) \; = \; \frac{f(\vec{s})}{\int_0^{\omega_i} f(t, s_{-i}) dt}.$$

We denote the corresponding distribution by $F_i(\cdot \mid s_{-i})$. The *conditional revenue curve* $B_i(\cdot \mid s_{-i})$ of bidder i is

$$B_i(s_i \mid s_{-i}) = v_i(\vec{s}) \int_{s_i}^{\omega_i} f_i(t \mid s_{-i}) dt. \tag{1}$$

The conditional revenue curve represents the expected revenue from letting bidder i win for price $v_i(s_i, s_{-i})$, given that the other signals are s_{-i}.[2]

We can now define the *conditional virtual value* of bidder i as

$$\varphi_i(s_i \mid s_{-i}) = -\frac{\frac{d}{ds_i} B_i(s_i \mid s_{-i})}{f_i(s_i \mid s_{-i})} = v_i(\vec{s}) - \frac{1 - F_i(s_i \mid s_{-i})}{f_i(s_i \mid s_{-i})} \cdot \frac{d}{ds_i} v_i(\vec{s}). \tag{2}$$

[2]Equation 1 uses the assumption that v_i is strictly increasing in s_i, so that the probability with which bidder i's value is at least $v_i(\vec{s})$, equals the probability $\int_{s_i}^{\omega_i} f_i(t \mid s_{-i}) dt$ that bidder i's signal is at least s_i.

For private values, the conditional virtual value simplifies to the more familiar form

$$\varphi_i(s_i \mid s_{-i}) = s_i - \frac{1 - F_i(s_i \mid s_{-i})}{f_i(s_i \mid s_{-i})}, \tag{3}$$

and for independent common values it takes the form

$$\varphi_i(s_i \mid s_{-i}) = v_i(\vec{s}) - \frac{1 - F_i(s_i)}{f_i(s_i)} \cdot \frac{d}{ds_i} v_i(\vec{s}). \tag{4}$$

Note there are three complications introduced by conditional virtual values, relative to standard virtual values in the independent private values model: first, the value $v_i(\vec{s})$ depends on the other signals; second, the inverse hazard rate is conditional on s_{-i}; and third, there is an extra dv_i/ds_i term.

We say that $F_i(\cdot \mid s_{-i})$ is *regular* if the conditional virtual value $\varphi_i(\cdot \mid s_{-i})$ is (weakly) increasing, and has monotone hazard rate, or *MHR*, if the inverse hazard rate $(1 - F_i(s_i \mid s_{-i}))/f_i(s_i \mid s_{-i})$ is (weakly) decreasing. A *monopoly price* is the signal s_i such that the conditional virtual value $\varphi_i(s_i \mid s_{-i})$ equals zero.

Following Myerson [1981], below we consider mechanisms that rank the bidders according to their conditional virtual values. In the independent private values model, regularity is sufficient for such a mechanism to be IC; however, the three complications introduced by conditional virtual values suggest that assumptions beyond regularity will be needed.

2.4. Settings

Here we present several settings of interest, which arise by imposing further natural assumptions on the single-parameter auction environment, and are extensively studied in the literature. One such assumption is regularity of the signal distribution; however regularity controls only the marginals, and with a joint signal/value distribution we may also need some kind of "global" restriction on the distribution — see below.

Matroid settings. Matroid settings arise by imposing structure on the feasible bidder subsets, which plays an analogous role to a "substitutes"-type property. The system of bidders and feasible subsets $(E, \mathcal{I})$ is a *matroid* if, in addition to being non-empty and downward-closed, the *exchange property* holds: for every $S, T \in \mathcal{I}$ such that $|S| > |T|$, there is some bidder $i \in S \setminus T$ such that $T \cup \{i\} \in \mathcal{I}$ (see, e.g., [Oxley 1992]). The exchange property can be thought of as a substitutes assumption. Examples of matroid settings include digital goods where $\mathcal{I} = 2^E$, k-unit auctions where $\mathcal{I}$ is all subsets of size at most k, and certain unit-demand matching markets corresponding to transversal matroids [Dhangwatnotai et al. 2010].

Settings with regularity or MHR. Settings in which the signal distribution F is regular (resp., MHR). That is, for every bidder i and signal profile s_{-i}, the conditional marginal distribution $F_i(\cdot \mid s_{-i})$ is regular (resp., MHR). The multivariate normal distribution in Example 2.3 is MHR, and thus also regular. Recall regularity is necessary even in the independent private values setting for the Myerson mechanism to be truthful without an additional ironing procedure.

Settings with affiliation. Settings which arise by imposing *affiliation* — a form of positive correlation — on the joint signal distribution. Affiliation was introduced by Milgrom and Weber [1982] and since then has become a standard assumption in the context of interdependence [Krishna 2010]. Signals are affiliated when observing a subset of high signals makes it more likely that the remaining signals are also high. Formally, for every signal profiles $\vec{s}, \vec{t} \in [0, \omega_i]^n$,

$$f(\vec{s} \vee \vec{t}) f(\vec{s} \wedge \vec{t}) \geq f(\vec{s}) f(\vec{t}),$$

where f is the density, $\vec{s} \vee \vec{t}$ is the component-wise maximum and $\vec{s} \wedge \vec{t}$ is the component-wise minimum (see also [Alon and Spencer 2008, FKG inequality]). The multivariate normal distribution in Example 2.3 is an affiliated distribution since all pairwise covariances are nonnegative [de Castro and Paarsch 2010]. For independent signals, the affiliation inequality holds with equality.

Symmetric settings. Symmetry involves assumptions on both valuation functions and signal distribution:

— For every bidder i, $v_i(\vec{s}) = v(s_i, s_{-i})$, where v is common to all bidders and symmetric in its last $n-1$ arguments;
— The joint density f is defined on support $[0, \omega]^n$ and is symmetric in all its arguments.

In a symmetric setting, bidders have the same conditional marginal densities, revenue curves and virtual values; however their values may be different (since their own signal is distinguished from others).

A *Milgrom-Weber setting* is a symmetric setting with affiliation [Milgrom and Weber 1982]. The weighted-sum values in Example 2.1 together with the symmetric multivariate normal distribution in Example 2.3 form a Milgrom-Weber setting.

Settings with a single crossing condition. A single crossing condition can be imposed on values or virtual values. In general, let $x_1(\cdot), \ldots, x_n(\cdot)$ be functions of the signals. A single crossing condition embodies the notion that bidder i's signal has a greater influence on x_i than on any other bidder's function x_j [Krishna 2010].[3] Formally, for every i and $j \neq i$, and for every $\vec{s}$,

$$\frac{\partial x_i}{\partial s_i}(\vec{s}) > \frac{\partial x_j}{\partial s_i}(\vec{s}). \tag{5}$$

Weaker versions of single crossing may require a non-strict inequality, or that the inequality hold only for $i, j, \vec{s}$ such that $x_i(\vec{s}) = x_j(\vec{s}) = \max_k\{x_k(\vec{s})\}$. Stronger versions may require the left-hand side of Equation 5 to be non-negative and the right-hand side non-positive (see, e.g., Lemma 5.3), or even that for every $\vec{s}$ such that $s_i > s_j$, $x_i(\vec{s}) > x_j(\vec{s})$ (see, e.g., Lemma 5.6). The weighted-sum values in Example 2.1 are weakly single crossing.

3. MECHANISM AND SOLUTION CONCEPTS

By the revelation principle, we focus without loss of generality on *direct mechanisms*, in which bidders directly report their private signals. An exception is the English auction, discussed below. We restrict attention to IC mechanisms and so make no distinction between reported and actual signals.

A (randomized) mechanism M consists of an *allocation rule* $x_i(\cdot)$ and a *payment rule* $p_i(\cdot)$ for every bidder i, where $x_i(\vec{s})$ is the probability over the internal randomness of the mechanism that bidder i wins given the other signals $\vec{s}$, and $p_i(\vec{s})$) is the expected payment of bidder i given s_{-i}. If M is deterministic, $x_i(\vec{s}) \in \{0, 1\}$ and $p_i(\vec{s})$ is the actual payment.

We assume risk-neutral bidders with quasi-linear utilities, i.e., given a mechanism and a signal profile $\vec{s}$, bidder i's effective utility is $x_i(\vec{s})v_i(\vec{s}) - p_i(\vec{s})$.

3.1. Solution Concepts

Mechanism design aims to define the rules of a game played by the bidders, such that a solution of the game has desirable properties like IC and IR. The solution concept — what constitutes a solution to the game — dictates the possibilities and impossibilities of

[3]The term "single crossing" comes from the fact that keeping all other signals fixed and varying only s_i, x_i as a function of s_i is "steeper" than x_j, so the two cross at most once.

mechanism design theory [Chung and Ely 2006]. We now describe three common solution concepts corresponding to three different types of equilibria.[4]

Definition 3.1 (*Ex post IC and ex post IR mechanism*). A mechanism is ex post IC and ex post IR if for every bidder i, true signal s_i, false report $\tilde{s}_i$, and signal profile s_{-i},

$$x_i(\vec{s})v_i(\vec{s}) - p_i(\vec{s}) \ \geq \ x_i(\tilde{s}_i, s_{-i})v_i(\vec{s}) - p_i(\tilde{s}_i, s_{-i}); \tag{6}$$
$$x_i(\vec{s})v_i(\vec{s}) - p_i(\vec{s}) \ \geq \ 0. \tag{7}$$

(Inequality 6 is the ex post IC condition and Inequality 7 is the ex post IR condition.)

In words: participating and truthtelling is an ex post equilibrium of the corresponding game, that is, it's a Nash equilibrium in the ex post stage of the game where private signals are common knowledge.

Definition 3.2 (*Dominant strategy IC mechanism*). A mechanism is dominant strategy IC if for every bidder i, true signal profile $\vec{s}$, and reported signal profile $\vec{r}$,

$$x_i(s_i, r_{-i})v_i(\vec{s}) - p_i(s_i, r_{-i}) \geq x_i(\vec{r})v_i(\vec{s}) - p_i(\vec{r}),$$

i.e., truthtelling is a dominant-strategy equilibrium of the corresponding game.

Definition 3.3 (*Bayesian IC and interim IR mechanism*). A mechanism is Bayesian IC and interim IR if for every bidder i, true signal s_i, and false report $\tilde{s}_i$,

$$\mathbb{E}_{s_{-i}}[x_i(\vec{s})v_i(\vec{s}) - p_i(\vec{s})] \ \geq \ \mathbb{E}_{s_{-i}}[x_i(\tilde{s}_i, s_{-i})v_i(\vec{s}) - p_i(\tilde{s}_i, s_{-i})];$$
$$\mathbb{E}_{s_{-i}}[x_i(\vec{s})v_i(\vec{s}) - p_i(\vec{s})] \ \geq \ 0.$$

That is: participating and truthtelling is a Bayesian-Nash equilibrium of the corresponding game in the interim stage, in which each individual knows his own signal but not the others.

3.2. Discussion of the Ex Post Solution Concept

The above definitions show that ex post is a weaker solution concept than dominant strategies (for which truthfulness holds for any reported signal profile), and a stronger one than Bayesian/interim (whose guarantees are in expectation over the true signal profile). We now briefly discuss our choice to focus on this intermediate solution concept. For additional discussion see, e.g., [Segal 2003; Milgrom 2004; Chung and Ely 2006; Krishna 2010].

Ex post vs. Bayesian. The solution concept most widely used in mechanism design theory is Bayesian Nash equilibrium [Chung and Ely 2006]. In practice, there are common auctions forms — the first-price and second-price auctions — which are Bayesian and not ex post. On the flip side, the Cremer-McLean mechanism has been criticized as impractical for lack of the ex post IR property. Bidders may regret participating, and the seller will need to extract large payments despite that. It also relies on common knowledge of the signal distribution.

Focusing on ex post mechanisms prevents these issues. Ex post IC and ex post IR are "no regret" properties — for any realization of the signals, bidders regret neither participating in the auction nor reporting their signals truthfully, even when all signals become publicly known. This makes the mechanism more robust (and thus closer to the computer science worst-case approach). To decide whether to participate and how to report, bidders do not have to know the signal distribution, only the signal support and the valuation functions. This is compatible with Wilson's doctrine of detail-free mechanisms that are robust to detailed knowledge of the distribution. Among its advantages, it saves transaction costs associated with learning about other bidders' distributions; it is also better for the seller who may be wary of using a Bayesian mechanism if she is not sure how well-informed bidders are.

[4] All IC and IR conditions hold in expectation over the internal randomness of the mechanism.

However there are two caveats. First, in settings such as the mineral rights model (Example 2.2), one can argue that a bidder's knowledge of his own valuation function $v(\vec{s}) = \mathbb{E}_{V \sim F_V}[V \mid \vec{s}]$ depends on his knowledge of the others' distributions — this is necessary for him to derive $v(\cdot)$ from the publicly known distribution F_V. In a model that crucially depends on bidders' knowledge of each other's distributions, and assuming the seller is aware that bidders are well-informed, there is less added robustness in an ex post solution over a Bayesian one.[5] (Note however that this issue does not arise in settings such as Example 2.1.) Second, it's not clear whether ex post is *necessary* for robustness; this question has been studied, e.g., in [Segal 2003; Bergemann and Morris 2005; Chung and Ely 2006].

Ex post vs. dominant strategies. For correlated or independent *private* values, dominant strategy IC and ex post IC coincide. For common values, the concept of dominant strategy IC guarantees an even stronger no regret property than the concept of ex post IC, since it does not depend on the other bidders reporting truthfully. For example, in the weighted-sum values case (Example 2.1), if bidder j under-reports his signal s_j, and bidder i somehow knows j's true signal, in an ex post mechanism i may potentially benefit by over-reporting his signal s_i, so that his true value is reflected by the mechanism.[6]

The following example demonstrates that the dominant strategy IC requirement may be too strong for a deterministic mechanism to extract non-trivial revenue. The example involves (by necessity) common and not private values.

Example 3.4. Two bidders compete for a single item. Their values are $v_1 = s_1 s_2$ and $v_2 = 0$, where $s_1, s_2 \in \{0, 1\}$. If one of the reported signals is 0, by ex post IR the mechanism gets zero revenue. For reported signal profile $\vec{r} = (1, 1)$, to achieve non-zero revenue the mechanism extracts from bidder 1 a payment bounded away from 0. However, if the true signal profile is $\vec{s} = (1, 0)$ but bidder 2 reports $r_2 = 1$, then bidder 1 is better off reporting $r_1 = 0$ untruthfully, in contradiction to dominant strategy IC.

3.3. The English Auction

We give a standard description of the English auction and its symmetric equilibrium in the appendix. Here it's sufficient to say that it is an ascending prive auction which works in "value space": in order to decide at which price to drop out of the auction, bidders must compare the current price to their postulated value based on their observations up to this point in the auction. This is unlike the other mechanisms we consider, which are direct revelation mechanisms that work in "signal space". Theorems 5.7 and 5.12 show that under certain conditions, the English auction provides an indirect implementation of the direct optimal mechanism.

4. RELATED WORK

Due to the richness of the literature on revenue guarantees of single-parameter auctions, a thorough discussion of related work is deferred to the appendix. In this section we briefly mention the main related results.

The first part of our work can be seen as a unification and generalization of many results appearing throughout the mechanism design literature. For independent private values, Myerson [1981] considers Bayesian IC and interim IR mechanisms, and establishes characterization, virtual surplus and optimal mechanism results. The first two among these apply to independent *common* values as well [Bulow and Klemperer 1996; Klemperer 1999]. Myerson's results cannot be extended to correlated signals. Cremér and McLean [1985; 1988] design an ex post IC but interim IR auction, which extracts full welfare in expectation when the correlation among signals has *full rank* (see also [McAfee and Reny 1992]).

[5]We thank the anonymous reviewers for this comment.
[6]We thank an anonymous reviewer for this example.

In their classic paper, Milgrom and Weber [1982] lay out a general model of interdependent values and develop the *linkage principle*, whice replaces revenue equivalence that no longer holds. They apply the linkage principle to rank the common auction formats (first-price, second-price, English and Dutch auctions) according to their expected revenue, when signals are affiliated and bids form a symmetric Bayesian-Nash equlibrium. Interestingly, for the English auction, the symmetric Bayesian-Nash equilibrium is actually an ex post equilibrium.

The ex post solution concept has generated much interest in the last decade. Most related to our work are the following papers, which study ex post mechanisms for interdependent values. Segal [2003] studies ex post IC and ex post IR mechanisms for selling multiple units of an item in the correlated private values model. Our results in this context can be seen as a generalization of Segal's results beyond multi-unit settings and beyond private values. A characterization result for ex post IC and ex post IR mechanisms in the interdependent values model can be found in Chung and Ely [2006] (see also [Lopomo 2000; Vohra 2011]). For a single item, Vohra [2011] states virtual surplus and optimal mechanism results, where the former is with respect to conditional virtual values as defined in Equation 2, and the latter is under the assumption that conditional virtual values are single-crossing. Vohra notes that for this result to be useful, one must identify restrictions on the distribution and valuation functions that would lead to single crossing. See also [Csapó and Müller 2013; Li 2013]. Jehiel at al. show impossibility results for ex post implementation in multi-parameter settings [Jehiel et al. 2006].

McAfee and Reny [1992] provide motivation for the study of the English auction. They conjecture that the English auction's prevelance in practice has to do with the need to perform well in a variety of circumstances, and call for formalizing the importance of robustness. Our work contributes to this goal. Bulow and Klemperer [1996] prove a virtual surplus result for the English auction with symmetric bidders under a strong regularity condition. Lopomo [2000] studies classes of mechanisms among which the English auction is optimal for the seller in single-item settings with interdependent values. Lopomo shows that in the Milgrom-Weber model with several additional assumptions (see Definition 5.8), the English auction with an optimal reserve price maximizes the expected revenue among "no-regret" mechanisms; we rederive a similar result in Corollary 5.12. Neeman [2003] studies settings in which the English auction generates nearly full surplus in revenue.

The computer science literature has studied computational considerationsa and near-optimal mechanisms in the correlated values setting. Papadimitriou and Pierrakos [2011] show a computational hardness result in the explicit model for correlated private values — finding the *deterministic*, ex post IC and ex post IR, optimal mechanism when there are at least three bidders is NP-hard. Computational hardness in the explicit model implies hardness in the oracle model as well. Their result does not extend to randomized mechanisms, which can be computed in the explicit model in polynomial time for all single-parameter domains, as well as unit-demand and additive multi-parameter domains [Dobzinski et al. 2011; Papadimitriou and Pierrakos 2011]. In the oracle model with correlated private values, Ronen [2001] designs the *lookahead auction* — a simple, deterministic, ex post IC and ex post IR mechanism, which guarantees a constant approximation to the optimal expected revenue. Dobzinski et al. [2011] build upon Ronen's work to design near-optimal mechanisms. They also show a gap between the best possible approximation by a deterministic mechanism and by a randomized one, when there are at least three bidders. This implies that additional assumptions are needed for a Myerson-like deterministic mechanism to be optimal (more on this in Section 5). When there are exactly two bidders, the optimal mechanism is deterministic, and can be computed in polynomial time [Dobzinski et al. 2011; Papadimitriou and Pierrakos 2011]. Ronen and Saberi [2002] show, in the oracle model, that a deterministic English auction cannot achieve an approximation ratio better than 3/4.

In the independent private values model, applications of Myerson's theory include simple near-optimal auctions [Hartline and Roughgarden 2009], prior-independent mechanisms [Dhangwatnotai et al. 2010], and prior-free mechanisms [Goldberg et al. 2006]. We study prior-independent mechanisms for interdependent values using techniques from Dhangwatnotai et al. [2010]; as we show in Section 6, our analysis necessarily diverges from theirs. Independently from and orthogonal to our work, Li [2013] shows a simple near-optimal auction for settings with interdependent values. She studies the VCG mechanism with monopoly reserves in matroid settings, where values satisfy a single crossing condition, and the valuation distribution satisfies the generalized monotone hazard rate condition. She shows that VCG with monopoly reserves extracts at least $1/e$ of the full surplus in expectation.

5. MYERSON THEORY FOR INTERDEPENDENT VALUES

The fundamental results of single-parameter optimal auction theory — Myerson's optimal mechanism and characterization results leading to it — do not carry over to interdependent settings. In this section we show that these results are at least partially recovered with small adaptations once we impose the ex post requirements.

The intuition behind this finding is as follows. The original proofs rely on signal independence so that both the probability x_i of winning and the expected payment p_i depend only on bidder i's reported signal, not on his true one. By switching from Bayesian IC and interim IR to ex post IC and ex post IR, we ensure the guarantees hold for any signal profile s_{-i}. Since we can now fix s_{-i}, rules x_i and p_i once again depend only on bidder i's reported signal, and so the independence assumption is no longer necessary.

5.1. Characterization and Expected Revenue of Ex Post Mechanisms

Here we see how far we can develop the theory with no assumptions whatsoever on the setting.

PROPOSITION 5.1 (CHARACTERIZATION). *For every interdependent values setting, a mechanism is ex post IC and ex post IR if and only if for every i, s_{-i}, the allocation rule x_i is monotone non-decreasing in the signal s_i, and the following payment identity and payment inequality hold:*

$$\begin{aligned} p_i(\vec{s}) \;&=\; x_i(\vec{s})v_i(\vec{s}) - \int_{v_i(0,s_{-i})}^{v_i(s_i,s_{-i})} x_i\left(v_i^{-1}(t \mid s_{-i}), s_{-i}\right) dt \\ &\quad - \left(x_i(0,s_{-i})v_i(0,s_{-i}) - p_i(0,s_{-i})\right); \qquad (8) \\ p_i(0,s_{-i}) \;&\leq\; x_i(0,s_{-i})v_i(0,s_{-i}). \qquad (9) \end{aligned}$$

PROOF. See Appendix. □

The payment identity and inequality imply that the allocation rule for every bidder determines the bidder's payment up to his expected payoff for a zero signal, and that this expected payoff must be non-negative. For private values, with a standard assumption of no positive transfers, the payment constraints simplify to the following identity

$$p_i(\vec{s}) = x_i(\vec{s})s_i - \int_0^{s_i} x_i\left(t, s_{-i}\right) dt. \qquad (10)$$

PROPOSITION 5.2 (REVENUE EQUALS VIRTUAL SURPLUS IN EXPECTATION). *For every interdependent values setting, the expected revenue of an ex post IC and ex post IR mecha-*

ALGORITHM 1: Myerson Mechanism for Interdependent Values

(1) Elicit signal reports $\vec{s}$ from the bidders.
(2) Maximize the conditional virtual surplus by allocating to the feasible set S with the highest non-negative conditional virtual value $\sum_{i \in S} \varphi_i(s_i \mid s_{-i})$, breaking ties arbitrarily but consistently.
(3) Charge every winner i a payment $p_i(\vec{s}) = v_i(s_i^*, s_{-i})$, where s_i^* is the threshold signal such that given the other signals s_{-i}, if i's signal were below the threshold he would lose.

nism equals its expected conditional virtual surplus up to an additive factor

$$\mathbb{E}_{\vec{s}}[\sum_i p_i(\vec{s})] = \mathbb{E}_{\vec{s}}[\sum_i x_i(\vec{s})\varphi_i(s_i \mid s_{-i})] - \sum_i \mathbb{E}_{s_{-i}}[(x_i(0, s_{-i})v_i(0, s_{-i}) - p_i(0, s_{-i}))] \tag{11}$$

Proof. See Appendix. □

Note that the additive factor is just minus the sum of the bidders' expected payoffs for zero signals. For private values this factor disappears.

Proposition 5.2 suggests that to optimize expected revenue, the best course of action is to maximize conditional virtual surplus pointwise. However the issue is monotonicity — even in the independent private values model, regularity is necessary for pointwise maximization to form a monotone allocation rule. We need more assumptions in more general models, as we derive below.

5.2. Optimal Mechanism for Correlated Private Values

In this section we focus on the correlated private values setting, with regularity and affiliation. A concrete example of such a setting, which also has the symmetry property, is values drawn from the multivariate normal distribution in Example 2.3. We begin with the following lemma, which is key to the analysis.

Lemma 5.3 (Single crossing of conditional virtual values). *For every regular affiliated signal distribution, and every private values setting, raising signal s_i weakly increases bidder i's conditional virtual value, and weakly decreases all other conditional virtual values.*

Proof. See Appendix. □

Now consider the allocation rule of the Myerson mechanism defined in Algorithm 1. The following lemma establishes its monotonicity in matroid settings.

Lemma 5.4 (Monotonicity). *For every regular affiliated signal distribution, and every matroid private values setting, maximizing conditional virtual surplus is monotone.*

Proof. Maximizing conditional virtual surplus can be implemented in a matroid environement by a greedy algorithm, which considers bidders in nonincreasing order of conditional virtual values, and adds them to the winning set if their conditional virtual value is nonnegative and if feasibility is maintained. By Lemma 5.3, raising signal s_i can only improve bidder i's ranking in the order of consideration, thus monotonicity holds. □

We our now ready to prove that the Myerson mechanism defined in Algorithm 1 is optimal.

Theorem 5.5 (Myerson mechanism is ex post IC, IR and optimal). *For every regular affiliated signal distribution, and every matroid private values setting, the Myerson* *mechanism is ex post IC, ex post IR, and optimal among all ex post IC and ex post IR mechanisms.*

Proof. By the characterization of ex post mechanisms (Proposition 5.1) applied to private values, for every bidder i it's sufficient to show that the allocation rule x_i is monotone in signal s_i, and that the payment identity $p_i(\vec{s}) = x_i(\vec{s})s_i - \int_0^{s_i} x_i(t, s_{-i})\, dt$ holds. Lemma 5.4 establishes monotonicity, and the payment identity holds by the following argument. The Myerson mechanism is deterministic and so either $x_i(\vec{s}) = 0$ or $x_i(\vec{s}) = 1$. In the former case, by monotonicity $x_i(t, s_{-i}) = 0$ for every $t \leq s_i$, so both sides of the identity are equal to zero. In the latter case, since s_i^* is bidder i's threshold signal, the right-hand side is

$$s_i - \int_0^{s_i} x_i(t, s_{-i})\, dt = s_i - (s_i - s_i^*) = s_i^*,$$

and for private values s_i^* is precisely the payment $p_i(\vec{s})$ charged by the Myerson mechanism.

It's left to show optimality. The expected revenue of an ex post IC and ex post IR mechanism is its expected virtual surplus (Proposition 5.2), and the Myerson mechanism maximizes virtual surplus for every signal profile. □

See Appendix for an example which demonstrates that maximizing conditional virtual surplus may be non-monotone in binary settings that are not matroid settings.

5.3. The English Auction for Correlated Private Values

Having established the properties of the Myerson mechanism in matroid settings, we now turn our attention to the English auction in single-item settings. We focus on symmetric settings, as without symmetry the English auction may not be optimal (see, e.g., [Ronen 2001]).

The main result in this section is a generalization of the following well-known result beyond the standard independent private values model — under regularity and symmetry, the second-price auction with optimal reserve coincides with the optimal Myerson mechanism. We show that for correlated private values, the same result holds with the English auction instead of the second-price one, provided that the optimal reserve price is set when only a single bidder remains in the auction. Recall from Section 3.3 that unlike the Myerson mechanism, the English auction is indirect. It's interesting to note that we get an indirect implementation of the optimal mechanism, that works directly in value space.

The proof uses the following stronger version of single crossing for virtual values (compared to Lemma 5.3).

Lemma 5.6 (Order of virtual values matches order of signals). *In the symmetric, affiliated private values setting with regularity, for every signal profile $\vec{s}$ and every two bidders for which $s_i \geq s_j$, the bidder with higher signal has higher conditional virtual value $\varphi_i(s_i \mid s_{-i}) \geq \varphi_j(s_j \mid s_{-j})$.*

Proof. See Appendix. □

Theorem 5.7 (English auction with optimal reserve is optimal). *In the symmetric, affiliated private values, single item setting with regularity, the English auction with the optimal reserve price is optimal among all ex post IC and ex post IR mechanisms.*

Proof. Consider the symmetric ex post equilibrium of the English auction. The winner is a bidder with the highest signal, who by Lemma 5.6 also has the highest virtual value. Now add an optimal reserve price to this auction. Since the reserve price is set after all the signals of the non-winner bidders' are revealed, it can prevent the item from being allocated to the potential winner when his conditional virtual value given the others' signals is negative. The resulting English auction with reserve is equivalent to the optimal Myerson mechanism (Algorithm 1). □

5.4. Optimal Mechanism and the English Auction for Interdependent Values

The results for correlated private values generalize to interdependent values under further assumptions. Recall why this setting is harder: conditional virtual values create two extra dependencies between bidders, which we need to address in order to establish monotonicity of an allocation rule.

In the previous section, to analyze the performance of the English auction we used Lemma 5.6, which states that the order of conditional virtual values coincides with the order of signals. In this section we begin by proving an analogue of this result. We adopt the approach of Lopomo [2000], and focus on the Milgrom-Weber model in which bidders are symmetric and have affiliated signals, with the following additional assumptions.

Definition 5.8 (*Lopomo assumptions*).

(1) MHR setting;
(2) Bidders with higher signals have higher values (strong single crossing of values);
(3) Bidders with higher signals have a lower sensitivity of their value to their own signal. Formally, for every signal profile $\vec{s}$ such that $s_i \geq s_j$,

$$\frac{\partial v_i}{\partial s_i}(\vec{s}) \leq \frac{\partial v_j}{\partial s_j}(\vec{s})$$

A concrete example of a Milgrom-Weber setting with Lopomo assumptions is the weighted-sum values in Example 2.1 together with the multivariate normal distribution in Example 2.3.

LEMMA 5.9 (ORDER OF VIRTUAL VALUES MATCHES ORDER OF SIGNALS). *In the Milgrom-Weber interdependent values setting with Lopomo assumptions, for every signal profile $\vec{s}$ and every two bidders for which $s_i \geq s_j$, the bidder with higher signal has higher conditional virtual value $\varphi_i(s_i \mid s_{-i}) \geq \varphi_j(s_j \mid s_{-j})$.*

PROOF. See Appendix. □

Lemma 5.9 implies that a strong form of single crossing holds for conditional virtual values (cf. Lemma 5.3), establishing monotonicity of the Myerson mechanism. We remark that there are also different (incomparable) assumptions that suffice to directly prove a form of single crossing. One example of alternative assumptions, which do not include bidder symmetry, is: affiliated signals, single crossing values and a *generalized MHR* condition [Li 2013, Assumptions], constant derivatives dv_i/ds_i. The analog of Lemma 5.9 in the independent private values setting holds when i.i.d. valuations are drawn from a regular distribution.

LEMMA 5.10 (MONOTONICITY). *For every Milgrom-Weber, interdependent values, matroid setting with Lopomo assumptions, maximizing conditional virtual surplus is monotone.*

PROOF. As in the proof of Lemma 5.4, it's enough to show that raising signal s_i can only improve bidder i's ranking in the greedy algorithm's order of consideration. This follows from Lemma 5.9, because the order of conditional virtual values is the same as the order of signals. □

As in the previous section, we now proceed to show that the Myerson mechanism defined in Algorithm 1 is optimal.

THEOREM 5.11 (MYERSON MECHANISM IS EX POST IC, IR AND OPTIMAL). *For every Milgrom-Weber, interdependent values, matroid setting with Lopomo assumptions, the Myerson mechanism is ex post IC, ex post IR, and optimal among all ex post IC and ex post IR mechanisms.*

PROOF. See Appendix. □

For completeness, we conclude this section with Lopomo's result on optimality of the English auction.

THEOREM 5.12 ([LOPOMO 2000], PROPOSITION 1). *For every Milgrom-Weber, interdependent values, single item setting with Lopomo assumptions, the English auction with the optimal reserve price is optimal among all ex post IC and ex post IR mechanisms.*

PROOF. See proof of Theorem 5.7, replacing Lemma 5.6 by Lemma 5.9 (conditional virtual values ordered like signals). □

5.5. Discussion of Assumptions

To establish truthfulness of the Myerson mechanism, that is, to show monotonicity of the virtual surplus maximizing allocation rule, we need some form of single crossing and regularity to apply to conditional virtual values. The former requirement stems from interdependence, whereas the latter applies to independent private values as well, but is circumvented by Myerson's *ironing* method.

Single crossing condition. Our assumptions are sufficient to establish single crossing: affiliation is sufficient for correlated private values, whereas for interdependent values a Milgrom-Weber setting with Lopomo assumptions is sufficient (there are alternative sufficient assumptions as well). An open question is to identify the *necessary* assumptions. The following example demonstrates what happens with lack of single crossing.

Example 5.13 (*Non-single crossing setting with regularity*). Consider a single item setting with weighted-sum values where parameter $\beta = 1$ (Example 2.1), and i.i.d. signals drawn from the regular but non-MHR variant of the *equal revenue distribution* $G(s) = 1 - s^{-2}$. Then plugging into Equation 4, the conditional virtual value of bidder i is $\sum_j s_j - (1 - G(s_i))/g(s_i) = s_i/2 + \sum_{j \neq i} s_j$. When bidder i's signal increases, his own conditional virtual value increases by $\Delta s_i/2$, while his competitors' conditional virtual values increase by Δs_i. Thus single crossing is violated, and the Myerson mechanism is not truthful.[7]

Regularity. Can ironing work in the interdependent values model? The method of ironing applies to conditional virtual values, and so the Myerson mechanism with ironing is well-defined for interdependent values. However, unlike the case of independent private values, the expected revenue of the Myerson mechanism with ironing can be strictly lower than its expected ironed conditional virtual surplus. Thus, maximizing the latter no longer guarantees that the maximum expected revenue is achieved. Technically, the issue is that within an ironed segment, while increasing a bidder's signal doesn't change his ironed conditional virtual value, it may change other bidders' ironed conditional virtual values, and so the allocation may change. Since the Myerson mechanism with ironing is deterministic, this observation is compatible with the gap between deterministic and randomized mechanisms identified in [Dobzinski et al. 2011; Papadimitriou and Pierrakos 2011].

6. PRIOR-INDEPENDENCE FOR INTERDEPENDENT VALUES

In this section we begin to develop a theory of prior-independence for interdependent values. We focus on a setting for which the results of the previous section apply, so that the Myerson mechanism provides a benchmark against which we can measure the expected revenue guarantees of prior-independent mechanisms. Our main result is, for this setting, a prior-independent mechanism that achieves a constant-factor approximation to the optimal expected revenue.

[7] in fact, the order of conditional virtual values is exactly the order of signals *reversed*; we can show that the optimal allocation rule is to simply pick a random winner.

ALGORITHM 2: The Single Sample Mechanism for Interdependent Values

(1) Elicit signal reports $\vec{s}$ from the bidders.
(2) Choose a *reserve bidder* uniformly at random, denote his signal by s_r.
(3) Place the feasible set of non-reserve bidders with highest signals in a "potential winners" set P. Break ties arbitrarily but consistently.
(4) Allocate to every bidder $i \in P$ such that $s_i \geq s_r$.
(5) Charge every winner i a payment $v_i(\max\{s_r, s_i^*\}, s_{-i})$, where s_i^* is the threshold signal such that given the signals of the other non-reserve bidders $s_{-\{i,r\}}$, if i's signal were below the threshold he would not belong to the potential winner set P.

6.1. Setting

Consider for simplicity a single item for sale. We assume that signals are independent and focus on the independent pure common value model. Within this model we focus on symmetric settings with $n \geq 2$ bidders, as is standard in the prior-independence literature. We further restrict attention to settings with Lopomo assumptions, so that the results of the previous section apply.

We also strengthen the first Lopomo assumption as follows. Let G be the distribution from which the i.i.d. signals are drawn, and let g be its density (G is the marginal distribution of every signal in the joint product distribution F). Given a fixed signal profile s_{-i}, bidder i's value is distributed according to the remaining random signal s_i. Its distribution, density and corresponding inverse hazard rate are

$$G_{|s_{-i}}(v_i) = G(v^{-1}_{i|s_{-i}});\ g_{|s_{-i}}(v_i) = g(v^{-1}_{i|s_{-i}})/\tfrac{d}{ds_i}v_i;\ \frac{1-G(s_i)}{g(s_i)} \cdot \frac{d}{ds_i}v_i(\vec{s}).$$

By the first Lopomo assumption, G is MHR. We require in addition that $G_{|s_{-i}}$ is MHR for every s_{-i}, i.e., that the above inverse hazard rate is (weakly) decreasing in s_i for every s_{-i}. A natural example in which this *strong MHR* assumption holds is as follows.

Example 6.1 (*Multilinear values*). Recall that a function is multilinear if it is separately linear in each one of its variables. Recall also that v_i is strictly increasing in s_i. So for every fixed signal profile s_{-i}, the multiplicand $\frac{d}{ds_i}v_i(\vec{s})$ is a positive constant. Since the multiplier $(1-G(s_i))/g(s_i)$ is decreasing by the MHR property of G, this implies that $G_{|s_{-i}}$ is MHR. Examples of multilinear values include the weighted-sum values in Example 2.1, signal products, etc.

We remark that for digital goods settings, our results hold more generally; e.g., we no longer need the Lopomo assumptions.

6.2. The Single Sample Mechanism for Interdependent Values

Algorithm 2 generalizes the prior-independent single sample mechanism of Dhangwatnotai et al. [2010] to interdependent values. Our main result in this section is that the generalized single sample mechanism is near-optimal. Let OPT denote the optimal expected revenue from the Myerson mechanism in Algorithm 1.

THEOREM 6.2 (SINGLE SAMPLE MECHANISM IS NEAR-OPTIMAL). *Let $n \geq 2$ and consider a single item for sale. In the symmetric, independent pure common value setting with Lopomo assumptions and a strong MHR assumption, the single sample mechanism in Algo-*

rithm 2 is prior-independent,[8] *and its expected revenue is a constant factor approximation to* OPT.

In the remainder of this section we prove Theorem 6.2. The proof relates the expected revenue of the single sample mechanism to the expected welfare, thus actually showing a stonger result of *effectiveness* as defined by Neeman [2003].

6.3. MHR Distributions

We present properties of MHR distributions that are used in the analysis of the single sample mechanism. We begin with an example that demonstrates the added difficulty due to interdependence over the independent private values case studied by Dhangwatnotai et al. [2010], explaining why the original analysis does not apply. The example shows that, unlike independent private values for which regularity suffices, for interdependent values a stronger MHR assumption is necessary to guarantee near-optimality of the single sample mechanism.

Example 6.3. Consider a digital goods setting with two bidders, whose i.i.d. signals are drawn from the equal revenue distribution $G(s) = 1 - (1/s)$, truncated to a finite range $[1, H]$, where H is an arbitrarily large constant. The bidders have weighted-sum values with $\beta = 1$ (Example 2.1), i.e., their pure common value is $s_1 + s_2$. The optimal expected revenue in this setting is at least $\mathbb{E}[s_1] + \mathbb{E}[s_1] \approx 2 \ln H$, by charging each bidder the signal of the other. The expected revenue of the single sample mechanism is $2\mathbb{E}[\min\{s_1, s_2\}]$, which is a constant that does not depend on H.

The gap between the expectation of the distribution and the expectation of the lower among two random samples is due to the equal revenue distribution's long tail; we now show that for MHR distributions this issue does not arise.

Lemma 6.4 (Lower among two samples from MHR distribution). *Let s, s' be two i.i.d. samples drawn from an MHR distribution G with density g, then*

$$\mathbb{E}[s \mid s \leq s'] \geq \frac{1}{2}\mathbb{E}[s].$$

Proof. Let $h(\cdot)$ be the hazard rate function of G and let $H(\cdot)$ be its cumulative hazard rate, i.e., $h(s) = g(s)/(1 - G(s))$, and $H(s) = \int_0^s h(z)dz$. By definition of the hazard rate function, $1 - G(s) = \exp(-H(s))$. Since G is MHR, $h(\cdot)$ is non-negative and (weakly) increasing, therefore $H(\cdot)$ is (weakly) increasing and convex [Dhangwatnotai et al. 2010].

$$\mathbb{E}[s \mid s \leq s'] = \int_0^\infty e^{-2H(s)} \geq \int_0^\infty e^{-H(2s)} \geq \frac{1}{2}\int_0^\infty e^{-H(s)} = \frac{1}{2}E[s],$$

where the first equality is by plugging in the distribution of the lower among two samples into $\mathbb{E}[x] = \int_0^\infty 1 - F(z)dz$, the first inequality is by convexity of H, and the second inequality is via integration by substitution. □

The above lemma is tight for the exponential distribution. We state the following simple lemma without proof.

Lemma 6.5 (Conditional MHR distribution). *Let signal s be randomly drawn from an MHR distribution G. For any threshold t, the distribution of s conditional on $s \geq t$ is also MHR.*

Additional properites of MHR and regular distributions appear in the appendix.

[8] We remark that the mechanism in Algorithm 2 is assumed to know the valuation functions. An open question is to design a mechanism, perhaps based on the English auction, for which this assumption can be dropped.

6.4. Main Lemma and Proof of Main Theorem

Let $c = 1/8e$, where e is the base of the natural logarithm.[9] Let s_1, s_2 be i.i.d. signals drawn from an MHR distribution G. Consider a single bidder with value $v(s_1, s_2)$ for some symmetric valuation function $v(\cdot, \cdot)$. Assume the strong MHR assumption holds, i.e., for every fixed s_1 or s_2, the corresponding distribution $G_{|s_1}$ or $G_{|s_2}$ of v is MHR.

LEMMA 6.6 (REUSING A SINGLE SAMPLE APPROXIMATES WELFARE). *If both signals are equal to the lower among two samples from G, the expected value decreases by a factor of at most c:*

$$\mathbb{E}_{s_1,s_2}[v(s_2, s_2)|s_2 \leq s_1] \geq c\mathbb{E}_{s_1,s_2}[v(s_1, s_2)].$$

PROOF. See Appendix. □

PROOF OF THEOREM 6.2 (SINGLE SAMPLE MECHANISM IS NEAR-OPTIMAL). The proof is by reduction to the above single bidder setting, and applying Lemma 6.6 — see Appendix for full proof. □

ELECTRONIC APPENDIX

The electronic appendix for this article can be accessed in the ACM Digital Library.

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