Why Prices Need Algorithms

TIM ROUGHGARDEN
and
INBAL TALGAM-COHEN
Stanford University

Understanding when equilibria are guaranteed to exist is a central theme in economic theory, seemingly unrelated to computation. In this note we survey our main result from [Roughgarden and Talgam-Cohen 2015], which shows that the existence of pricing equilibria is inextricably connected to the computational complexity of related optimization problems: demand oracles, revenue-maximization and welfare-maximization. We demonstrate how this relationship implies, under suitable complexity assumptions, a host of impossibility results. We also suggest a complexity-theoretic explanation for the lack of useful extensions of the Walrasian equilibrium concept: such extensions seem to require the invention of novel polynomial-time algorithms for welfare-maximization.

Categories and Subject Descriptors: F.2 [**Theory of Computation**]: Analysis of Algorithms and Problem Complexity; J.4 [**Computer Applications**]: Social and Behavioral Sciences—*Economics*

General Terms: Algorithms, Economics, Theory

Additional Key Words and Phrases: Market Equilibrium, Computational Game Theory, Equilibrium Computation, Market Design, Complexity

1. INTRODUCTION

Computational complexity has already had plenty to say about the *computation* of economic equilibria (for example, [Fischer et al. 2006; Chen et al. 2009; Chen et al. 2009; Daskalakis et al. 2009; Papadimitriou and Wilkens 2011]). The primary theme of [Roughgarden and Talgam-Cohen 2015], surveyed in this note, is that complexity can also be used to study equilibrium *existence*. In particular, it turns out that non-existence results can be derived from the computational intractability of related optimization problems, under widely-believed complexity assumptions like P $\neq$ NP. We explore this theme in the classic setting of market equilibria for markets with indivisible items and quasi-linear consumers.

We begin by stating our main result for Walrasian equilibria (the leading notion of market equilibrium in our context), which we explain below in Section 2, and generalize in Section 3, before we conclude with open problems:

Theorem 1.1. *A necessary condition for the guaranteed existence of a Walrasian equilibrium in markets with valuations from class $\mathcal{V}$ is that demand queries for $\mathcal{V}$ given item prices are as hard computationally as welfare maximization for $\mathcal{V}$.*

(By "as hard as computationally" we refer to the existence of a polynomial-time Turing reduction from the latter to the former.)

Theorem 1.1 establishes a link between a purely economic question – existence

Authors' addresses: `tim@cs.stanford.edu`, `inbaltalgam@gmail.com`

of equilibrium – and a purely algorithmic one. We restate it as its contrapositive, which is useful for proving nonexistence results as demonstrated below:

COROLLARY 1.2. *If, under standard complexity assumptions, welfare maximization in markets with valuations from class $\mathcal{V}$ cannot be reduced to demand queries for $\mathcal{V}$ given item prices, then the existence of a Walrasian equilibrium is not guaranteed.*

2. WALRASIAN EQUILIBRIUM

Consider a standard market model, with n consumers and m items. Each consumer i has a valuation function v_i, which is a map from bundles of items to their value in $\mathbb{R}_{\geq 0}$. Consumers have quasi-linear utilities (their utility from a bundle is their value for it minus the bundle's price). While a valuation in general requires 2^m numbers to represent, it is natural to assume that consumers either have a succinctly represented valuation or oracle access to it.

A Walrasian equilibrium, a fundamental economic concept dating back to the work of [Walras 1874], is a pair consisting of (1) an allocation of the indivisible items to the consumers, and (2) a price for each item, such that the following two conditions hold. These conditions are both necessary and sufficient for stability of the market:

(1) Every consumer is allocated a bundle that is in his demand, i.e., maximizes his utility given the prices;
(2) The revenue is maximized given the prices, i.e., all items with non-zero prices are allocated.

The first welfare theorem establishes that the allocation in every Walrasian equilibrium is welfare-maximizing. Moreover, it is welfare maximizing among all possible *fractional* allocations. In fact, the welfare maximization problem for a given market can be written as a linear program, called the *configuration LP*, whose optimal solution is an integral allocation if and only if a Walrasian equilibrium exists for this market (see, e.g., [Blumrosen and Nisan 2007]). This leads to the question: for which class of markets is a Walrasian equilibrium guaranteed to exist?

In their seminal work, [Kelso and Crawford 1982] introduced the valuation class of gross substitutes. There are many alternative definitions for gross substitutes (see [Paes Leme 2014]), but the intuition is that the items are treated by the consumer like coffee and tea than like coffee and dessert – they are not complementary in his view, and so if the price of one (coffee) rises, more of the other (tea) will be in demand. An example of gross substitutes valuations are unit-demand valuations, where the consumer has no value for more than one item (he will not drink both coffee and tea). Gross substitutes are important since Walrasian equilibria are guaranteed to exist in markets with such valuations. Moreover, the other direction partially holds as well – any class of valuations other than gross substitutes which includes unit-demand valuations does not guarantee existence of a Walrasian equilibrium [Gul and Stacchetti 1999; Milgrom 2000].

As the gross substitutes condition is quite stringent and unlikely to hold in practice, research did not stop at this, and proceeded to study classes of valuations that exclude unit-demand valuations in a relatively ad hoc fashion [Parkes and Ungar

2000; Sun and Yang 2006; Ben-Zwi et al. 2013; Candogan et al. 2014; Candogan and Pekec 2014; Sun and Yang 2014; Teytelboym 2014; Candogan et al. 2015]. Another research direction pursued by [Bikhchandani and Ostroy 2002] and others allows bundle prices rather than item prices, as well as personalization of prices, and studies the resulting generalizations of Walrasian equilibria called pricing equilibria (see Section 3).

2.1 Our Results for Walrasian Equilibrium

Let $\mathcal{V}$ be a class of valuations (e.g., unit-demand valuations). There are two algorithmic problems related to Walrasian equilibria in markets with valuations from $\mathcal{V}$:

(1) Welfare maximization: A social planner gets as input n consumer valuations, and needs to output a welfare-maximizing allocation.
(2) Utility maximization, also known as answering demand queries: A consumer gets as input a vector of item prices, and needs to output a bundle of items that maximizes his utility given these prices.

Recall that our main result for Walrasian equilibrium, stated in its contrapositive, is that if welfare maximization cannot be reduced to demand queries under standard complexity assumptions, then the existence of a Walrasian equilibrium in markets with valuations from $\mathcal{V}$ is not guaranteed (Corollary 1.2). The proof is short and leans on the configuration LP and its solution via the ellipsoid method [Nisan and Segal 2006]. Our result can be applied to rule out guaranteed existence of Walrasian equilibria for candidate classes of markets.

For example, imagine a consumer who has different values for different kinds of dessert, and aggregate values for bundles of desserts up to a capacity (or budget) b on the total value he can extract from dessert. Instantiate $\mathcal{V}$ with the class of *budget-additive* valuations, which captures such consumers. The goal when solving a demand query is then to maximize value within the capacity at minimum price, and this reduces to the knapsack problem. The goal of welfare-maximization is to fit maximum total value within the n consumer capacities, and this is as hard as the bin packing problem. While knapsack and bin packing are both NP-hard, the former is only weakly so while the latter is strongly so [Garey and Johnson 1979]. Thus, assuming that values and capacities are all polynomially-bounded integers, if $\mathrm{P} \neq \mathrm{NP}$ then:

$$\text{Bin packing} \propto \text{welfare maximization} \not\propto \text{demand query} \propto \text{knapsack}.$$

Applying Corollary 1.2 we conclude that there exists a market with budget-additive valuations and no Walrasian equilibrium.

This example demonstrates our approach to non-existence results, which stands on the mature understanding of computational complexity and is arguably more systematic and elucidating than previous approaches, with an added dependence on complexity assumptions.

3. GENERAL PRICES AND PRICING EQUILIBRIUM

We have mentioned three levels of generality in pricing so far:

(1) Item prices (a vector of m prices);

(2) Bundle prices (a vector of $2^m - 1$ prices);
(3) Personalized bundle prices (n vectors of $2^m - 1$ prices).

Notice that mathematically speaking, a pricing function is identical to a valuation, since it is also a function from bundles to $\mathbb{R}_{\geq 0}$. Just as there are many natural and well-studied classes of valuations, there are many classes of pricings that we can consider beyond the above three. The generalization of the Walrasian equilibrium notion to allow for personalized pricings from such a class is called a pricing equilibrium [Nisan and Segal 2006], and as in the original definition, two conditions must hold:

(1) Every consumer's allocation maximizes his utility given his personal pricing;
(2) The allocation maximizes revenue given the pricings.

As [Nisan and Segal 2006] show, the welfare theorems continue to hold for pricing equilibria. We remark that by Condition (2), the market cannot be stabilized by setting consumer i's price for all bundles not allocated to him to be ∞, since then the revenue would also have to be ∞. Condition (2) also gives rise to a third algorithmic problem in addition to welfare maximization and demand queries: In the *revenue maximization* problem, a seller gets as input n consumer pricings, and needs to output a revenue-maximizing allocation.

One could expect that since we have allowed more general pricing, the existence of pricing equilibria will be guaranteed for many more markets than for Walrasian equilibria. This is indeed true, but in an uninteresting way. It is straightforward to show the existence of a degenerate pricing equilibrium as follows:

PROPOSITION 3.1. *Consider a valuation class $\mathcal{V}$ and an identical pricing class $\mathcal{P} = \mathcal{V}$. For every market with valuations from $\mathcal{V}$ there exists a pricing equilibrium with pricings from $\mathcal{P}$, where the pricing for each consumer is simply his valuation.*

What makes this equilibrium uninteresting, whereas Walrasian equilibrium is a truly remarkable notion? A Walrasian equilibrium employs prices that are "simple" in three respects: they are anonymous (different consumers face the same prices); they are succinct in comparison to the class of valuations they stabilize (they are m-dimensional, whereas the dimension of the gross substitutes valuation class is exponential in m); and they make the *verification* of the equilibrium tractable. For example, given an alleged Walrasian equilibrium, it is trivial for the seller to verify the revenue-maximization condition by checking whether all unsold items have zero price; but for the above degenerate equilibrium, revenue maximization is as hard as welfare maximization. The main question is then, what other simple and meaningful pricing equilibria exist? Or more accurately, why are no such equilibria known to date?

3.1 Our Results for Pricing Equilibria

We give a brief overview of our results for pricing equilibria; details appear in [Roughgarden and Talgam-Cohen 2015].

First, the methodology we introduced for Walrasian equilibria can be utilized to show the non-existence of interesting pricing equilibria for many markets that seem like natural candidates for positive results. For example, unit-demand valuations guarantee an equilibrium with anonymous item prices; what if we generalize to "pair-demand" valuations, would they guarantee an equilibrium with anonymous prices on pairs of items? A corollary of one of our results (a parallel of Theorem 1.1 that applies to pricing equilibria with anonymous pricing) is that they will not, at least unless NP $\subseteq$ coNP.

Moreover, our methodology provides an explanation to the dearth of useful extensions of the Walrasian equilibrium concept, by linking the existence of such extensions to algorithmic progress on the welfare-maximization problem. In other words, the challenge of finding a novel polynomial-time algorithm for the welfare-maximization problem, beyond solving the well-known configuration LP, poses an algorithmic barrier to such results.

4. CONCLUSION

The well-studied problem of proving or disproving the guaranteed existence of pricing equilibria seems to have nothing to do with computation. We demonstrate in [Roughgarden and Talgam-Cohen 2015] that computational complexity offers numerous insights into the problem, and provides general techniques for proving impossibility results.

Many questions remain, beginning with the design of new welfare maximization methods. Does a "converse" of Theorem 1.1 hold, i.e., will non-trivial pricing equilibria emerge from such methods? Also, is our methodology applicable beyond economies with quasi-linear utilities (see [Segal 2007])? More generally, in this work we link an economic property of gross substitutes (guaranteed equilibrium existence) to their algorithmic properties, does such a link exist for other good economic properties of this valuation class?

REFERENCES

Ben-Zwi, O., Lavi, R., and Newman, I. 2013. Ascending auctions and Walrasian equilibrium. Working paper.

Bikhchandani, S. and Ostroy, J. M. 2002. The package assignment model. *Journal of Economic Theory 107,* 2, 377–406. Following 1998 technical report.

Blumrosen, L. and Nisan, N. 2007. Combinatorial auctions. In *Algorithmic Game Theory*, N. Nisan, T. Roughgarden, E. Tardos, and V. V. Vazirani, Eds. Chapter 11.

Candogan, O., Ozdaglar, A., and Parrilo, P. 2014. Iterative auction design for graphical valuations part II: General graphs. In submission.

Candogan, O., Ozdaglar, A., and Parrilo, P. 2015. Iterative auction design for tree valuations. To appear in Operations Research.

Candogan, O. and Pekec, S. 2014. Efficient iterative auctions for multi-featured items: A network flow approach. In submission.

Chen, X., Dai, D., Du, Y., and Teng, S.-H. 2009. Settling the complexity of Arrow-Debreu equilibria in markets with additively separable utilities. In *Proceedings of the 50th Symposium on Foundations of Computer Science*. 273–282.

Chen, X., Deng, X., and Teng, S.-H. 2009. Settling the complexity of computing two-player Nash equilibria. *Journal of the ACM 56,* 3.

Daskalakis, C., Goldberg, P. W., and Papadimitriou, C. H. 2009. The complexity of computing a Nash equilibrium. *SIAM Journal on Computing 39,* 1, 195–259.

Fischer, F. A., Holzer, M., and Katzenbeisser, S. 2006. The influence of neighbourhood and choice on the complexity of finding pure Nash equilibria. *Information Prrocessing Letters 99,* 6, 239–245.

Garey, M. R. and Johnson, D. S. 1979. *Computers and Intractability: A Guide to the Theory of NP-Completeness.*

Gul, F. and Stacchetti, E. 1999. Walrasian equilibrium with gross substitutes. *Journal of Economic Theory 87*, 95–124.

Kelso, A. S. and Crawford, V. P. 1982. Job matching, coalition formation, and gross substitutes. *Econometrica 50,* 6, 1483–1504.

Milgrom, P. R. 2000. Putting auction theory to work: The simultaneous ascending auction. *Journal of Political Economy 108,* 2, 245–272.

Nisan, N. and Segal, I. 2006. The communication requirements of efficient allocations and supporting prices. *Journal of Economic Theory 129*, 192–224.

Paes Leme, R. 2014. Gross substitutability: An algorithmic survey. Working paper.

Papadimitriou, C. H. and Wilkens, C. A. 2011. Economies with non-convex production and complexity equilibria. In *Proceedings of the 12th ACM Conference on Economics and Computation.* 137–146.

Parkes, D. C. and Ungar, L. H. 2000. Iterative combinatorial auctions: Theory and practice. In *Proceedings of the 17th AAAI Conference on Artificial Intelligence*. 74–81.

Roughgarden, T. and Talgam-Cohen, I. 2015. Why prices need algorithms. In *Proceedings of the 16th ACM Conference on Economics and Computation.* 19–36.

Segal, I. 2007. The communication requirements of social choice rules and supporting budget sets. *Journal of Economic Theory 136,* 1, 341–378.

Sun, N. and Yang, Z. 2006. Equilibria and indivisibilities: Gross substitutes and complements. *Econometrica 74,* 5, 1385–1402.

Sun, N. and Yang, Z. 2014. An efficient and incentive compatible dynamic auction for multiple complements. *Journal of Political Economy 122,* 2, 422–488.

Teytelboym, A. 2014. Gross substitutes and complements: a simple generalization. *Economics Letters 123,* 2, 135–138.

Walras, L. 1874. *Elements of Pure Economics.* Allen and Unwin. Published in 1954.