Competitive Equilibrium with Indivisible Goods & Generic Budgets

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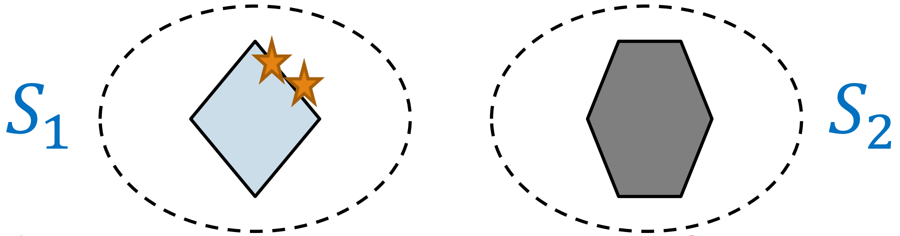
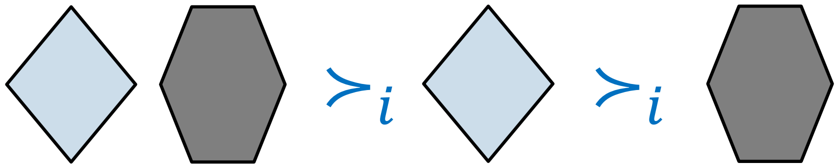
Model

Fisher market with m indivisible items



n players, each with strict monotone preference $\succ_i$ and budget b_i

- "No value for money"
- Wlog $\sum_i b_i = 1$



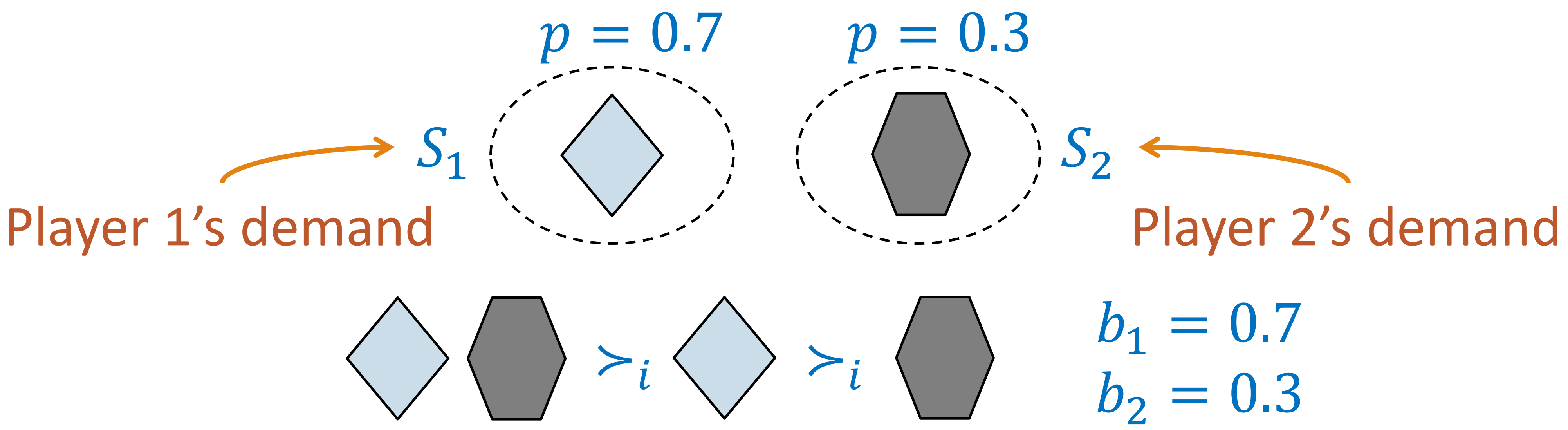
$b_1 = 0.7$
$b_2 = 0.3$

Allocation $\mathcal{S}$ = partition of all items among players

Competitive Equilibrium (CE) $(\mathcal{S}, \boldsymbol{p})$

Allocation $\mathcal{S}$ and item prices $\boldsymbol{p}$ s.t. each player gets her demand

- Demand = most preferred among bundles she can afford



Our High-Level Goal

We study existence and fairness properties of CE in Fisher markets with indivisible items and possibly unequal budgets

Motivation I: Existence

Example: 1 item, 2 players, $b_1 = b_2 = 0.5$ → no CE!

- $p \leq 0.5$ both demand
- $p > 0.5$ neither demand

(Many early works stop here)

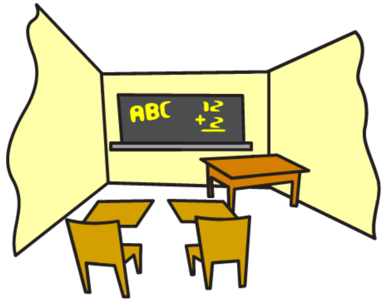
But what if budgets are generic? $b_1 = 0.5 + \epsilon, b_2 = 0.5 - \epsilon$ → CE!

Q1: When do generic budgets guarantee CE existence?

Related Work

[Budish’11]: Approximate CE exists with almost equal budgets, even for non-monotone preferences

- Course allocation application
- Approximate = may need to add a few seats to each class



Other work on CE existence with indivisible goods:

- One divisible good [Broome’72, Svensson’83, Maskin’87, Alkan-Demange-Gale’91, …], house allocation (unit-demand) setting [Shapley-Scarf’74, Svensson’84, …], relaxed equilibrium notions [Starr’69, Arrow-Hahn’71, Dierker’71, …], continuum of traders [Mas-Colell’95, …]

Results I: Existence

Q1: When do generic budgets guarantee CE existence?

Main result: Sufficient conditions for existence for two players with additive preferences:

1. Almost-equal budgets, or
2. Existence of a "proportional" allocation, or
3. Symmetric preferences

Additional (non-)existence results in paper

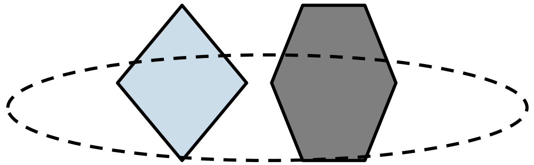
Motivation II: Fairness

Background: CE guarantees Pareto efficient allocation

- (No other allocation more preferred by players whose bundle changes)

For divisible items, CE also guarantees fair allocation

1. Fair-share
   - Player i prefers her bundle to a b_i-fraction of all items



$b_i = 0.7$

2. (For equal budgets, envy-freeness)

[Budish'11] generalizes these fairness notions to indivisible items, CE with almost-equal budgets

Motivation II: Fairness

Q2: What are fairness guarantees of a CE with unequal budgets?

Q2': What is a "fair" allocation of indivisible items when players have unequal entitlements?

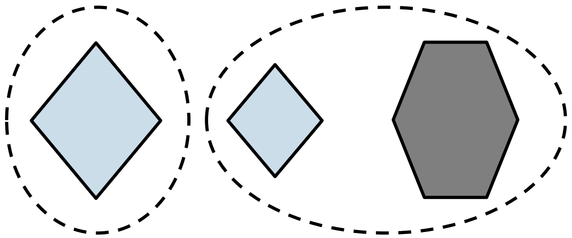
Example of unequal entitlements:

- In course allocation, 1st- versus 2nd-year MBA students

Results II: Fairness

We define a generalization of fair-share to unequal entitlements

Main fairness result: CE with unequal budgets guarantees that each player prefers her bundle to her generalized fair-share



Divide

Let other player choose

Generalizations of fair-share:

- [Budish'11]: maximin-share
- Our generalization: "ℓ-out-of-d" maximin-share

Results II: Fairness

We also define a player's "truncated-share"

Our CE existence results for 2 additive players guarantee:

- Each player prefers her bundle to her truncated-share

Related work:

- [Brams-Taylor'96, Bouveret-et-al'16, Farhadi-et-al'17, …]

High-Level Goal Revisited

We study existence and fairness properties of CE in Fisher markets with indivisible items and possibly unequal budgets

Results:

1. Show settings of interest (embracing non-general preferences) where generic budgets guarantee existence
2. Show that CE guarantees (in fact helps define) fairness for players with unequal entitlements

One take-away: Model + fairness objective merit more study

Results in More Detail

1. EXISTENCE
2. FAIRNESS

Recall Main Existence Result

Theorem: Sufficient conditions for CE existence for 2 additive players

1. Almost-equal budgets $b_2 = b_1 - \epsilon$, or
2. Existence of a proportional allocation, or
3. Identical preferences and generic budgets

"Budish-style" fairness

Players in direct competition

Additional (non-)existence results in paper

Now: Why additive preferences?

Non-Existence for General Preferences

Theorem: There exists 2 (non-additive) players and 5 items such that for an open interval of budgets, no CE exists

Proof idea: Mimic CE non-existence in the quasi-linear model

- 2 players and 2 items, 1^{st} player views as complements, 2^{nd} as substitutes
- 3 extra items mimic money

Proposition: For 2 players, 4 items and generic budgets, a CE exists

Additive Preferences, Combination Prices

Additive preference: Player i has value $v_{i,j}$ for item j

- Value of bundle is sum of item values

2 additive players induce (α, β)-combination prices

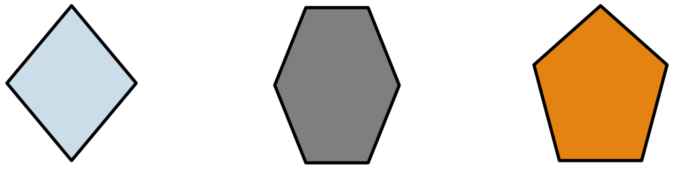
Example (unnormalized):

$v_1 = 3$ $v_1 = 5$ $v_1 = 2$

$v_2 = 1$ $v_2 = 4$ $v_2 = 7$

$\alpha = 2$

$\beta = 1$



$p = 7$ $p = 14$ $p = 11$

$p(T) = \alpha v_1(T) + \beta v_2(T)$
(additivity comes in handy)

Key Lemma

Lemma: For 2 additive players, if there exist budget-exhausting combination prices $\boldsymbol{p}$ for a Pareto optimal allocation $\mathcal{S}$ → then $(\mathcal{S}, \boldsymbol{p})$ is a CE

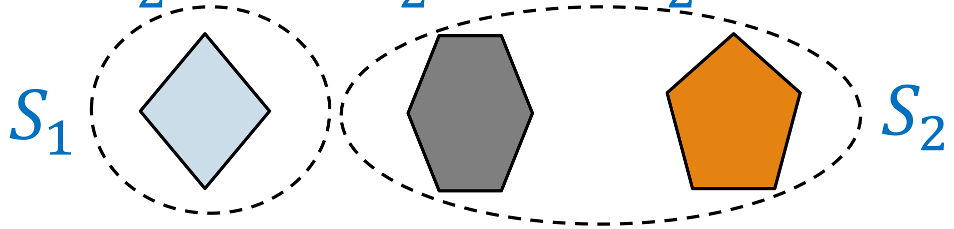
Corollary: 2nd Welfare Theorem

$v_1 = 3$ $v_1 = 5$ $v_1 = 2$

$v_2 = 1$ $v_2 = 4$ $v_2 = 7$

$b_1 = 7$

$b_2 = 25$



$p = 7$ $p = 14$ $p = 11$

Pareto optimal

Budgets exhausted

Recall Main Existence Result

Theorem: Sufficient conditions for CE existence for 2 additive players

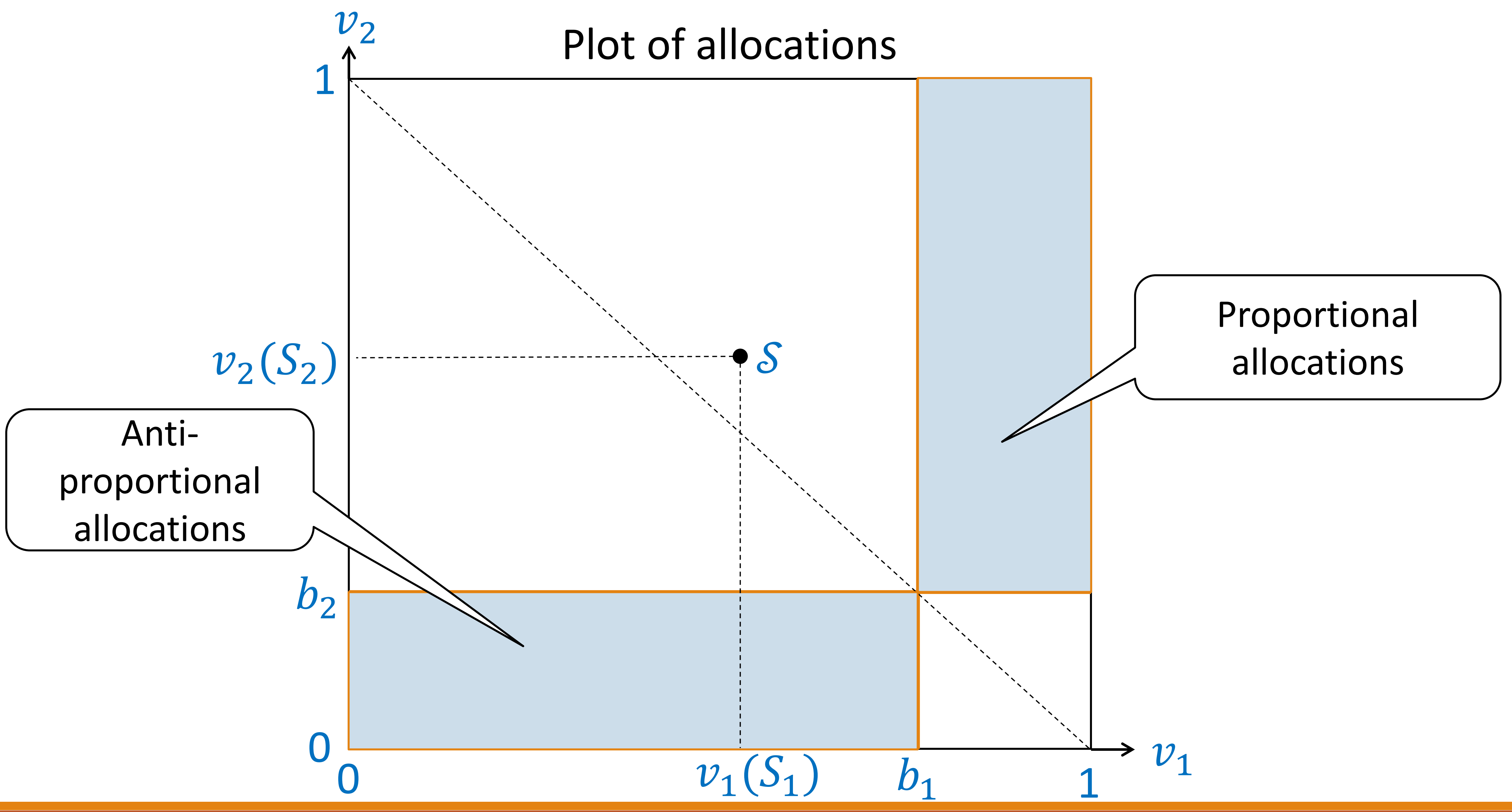
1. Almost-equal budgets $b_2 = b_1 - \epsilon$, or
2. Existence of a proportional allocation, or
3. Identical preferences and generic budgets

Assume wlog normalized values $v_i(\text{all items}) = 1$

"Proportional" = fair-share allocation when preferences are cardinal

- For player i, $v_i(S_i) \geq b_i$
- Extends immediately to indivisible items

Plot of allocations

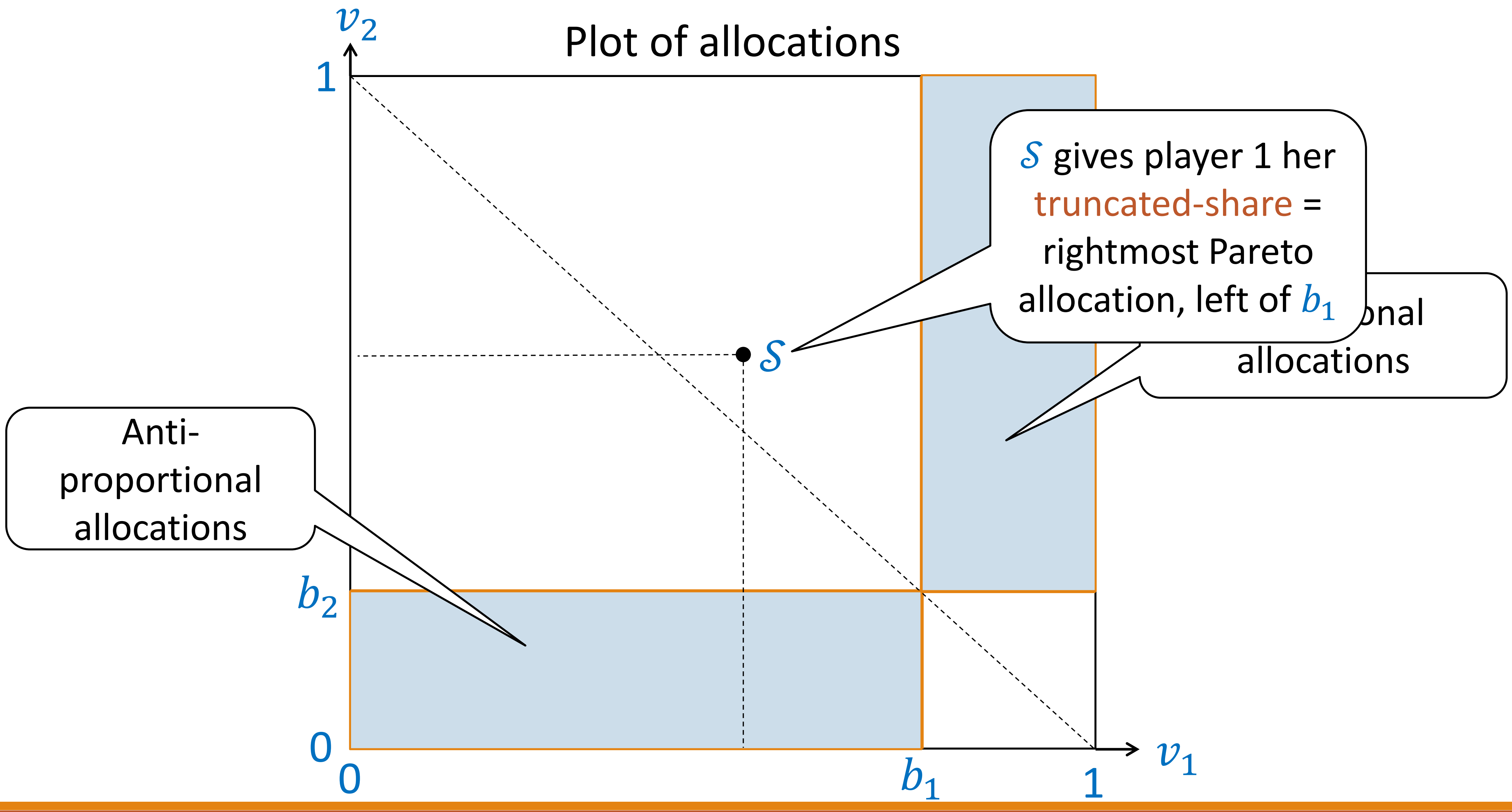


Proof that Proportionality is Sufficient

Recall key lemma: For 2 additive players, if there exist budget-exhausting combination prices $\boldsymbol{p}$ for a Pareto optimal allocation $\mathcal{S}$ → then $(\mathcal{S}, \boldsymbol{p})$ is a CE

Proportionality inequalities imply existence of budget-exhausting combination prices and thus of a CE

Plot of allocations



Recall Main Fairness Result

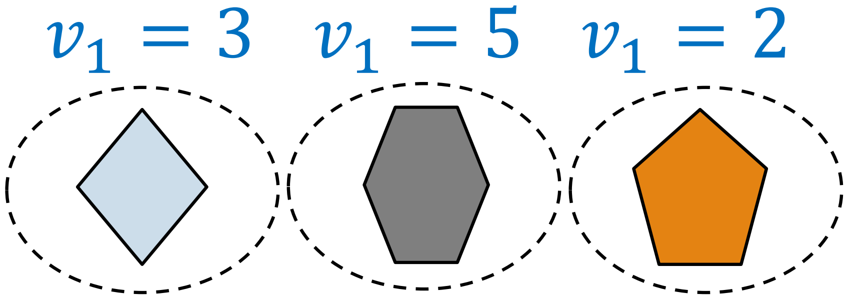
Proposition: Every CE guarantees that each player i prefers her bundle to her "ℓ-out-of-d" maximin-share where $\ell/d \leq b_i$

(Our CE existence results for 2 additive players also guarantee that each player prefers her bundle to her truncated-share)

ℓ-out-of-d Maximin Share

Definition: Most preferred bundle a player can guarantee by dividing the items to d parts, and letting the other choose all but ℓ parts

Example: 1-out-of-3 maximin share of player 1



Divide

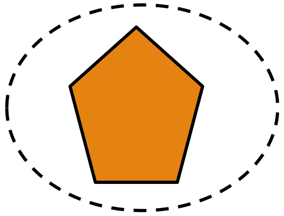
Let other player choose

ℓ-out-of-d Maximin Share

Proposition: In CE, for every $\ell/d \leq b_i$, player i prefers her bundle to her ℓ-out-of-d maximin share

Example: Let $b_1 = 5/13$, then player 1 gets at least her 1-out-of-3 maximin share (since $1/3 \leq 5/13$)

$v_1 = 3 \quad v_1 = 5 \quad v_1 = 2$



Recap

We study existence and fairness properties of CE in Fisher markets with indivisible items and possibly unequal budgets

Results:

1. Show settings of interest (embracing non-general preferences) where generic budgets guarantee existence
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One take-away: Model + fairness objective merit more study

Open Questions

Unconditional existence of CE for 2 additive players with generic budgets?

Alternatively, existence of “fair” allocation given entitlements?

Practical mechanisms / heuristics for finding CE / fair allocation?

Beyond additive

Thanks for listening!