Oblivious Rounding
and the
Integrality Gap

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Setting: A Maximization Problem (V, X)

IN GENERAL

X = set of feasible solutions

V = set of linear objective functions

X, V are sets of non-negative real vectors

EXAMPLE: MAX-CUT IN COMPLETE WEIGHTED GRAPHS OF n VERTICES

X = set of cuts

V = set of edge weight functions

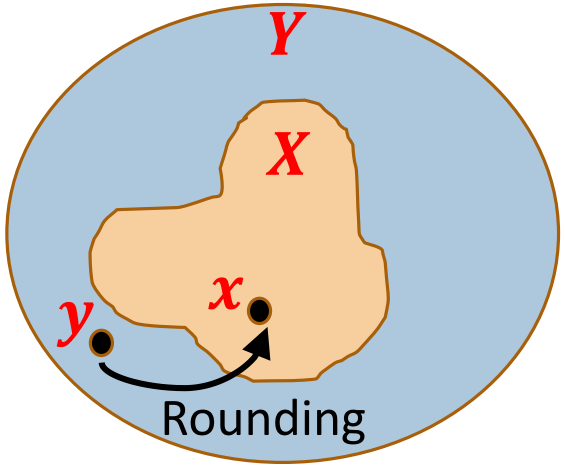
X, V are sets of vectors of dimension $\binom{n}{2}$ (vectors in X are $\{0,1\}$-vectors)

Given $v \in V$, find $x \in X$ that maximizes $v \cdot x$

Oblivious Rounding

Classic approach to hard maximization problem (V, X):

1. Relax (V, X) to (V, Y) where $X \subset Y$ and Y is fractional
2. Given $v \in V$ find $y \in Y$, guarantees $v \cdot y$
3. Round y to $x \in X$



The approximation ratio of the rounding is $\frac{v \cdot x}{v \cdot y}$ (in the worst case)

If step 3 does not use v, we call the rounding "oblivious"*

*Not to be confused with [Young'95]

Examples from the Literature

OBLIVIOUS ROUNDING

Threshold rounding for vertex cover
- [Hochbaum'82]

Randomized rounding for set cover
- [Raghavan-Thompson'87]

Random hyperplane rounding for max-cut
- [Goemans-Williamson'95]

Welfare maximization for submodular valuations
- [Feige'09, Feige-Vondrak'10]

NON-OBLIVIOUS ROUNDING

Rounding of SDPs for CSP
- [Raghavendra-Steurer'09]

Facility location
- [Li'13]

Welfare maximization for gross substitutes valuations
- [Nisan-Segal'06]

Main Question

For which problems and relaxations can we expect oblivious rounding to give a good approximation ratio?

A question about information
- Rounding not restricted to be in polynomial time

Answer useful for:
- Algorithm designers
- Mechanism designers (details in a few slides)

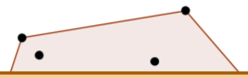
Main Result
& Application

Main Result

[Informal] The approximation ratio of the best oblivious rounding scheme for a given relaxation = the integrality gap of the problem's closure

The closure of problem (V, X) is $(C(V), X)$

- where $C(V)$ is the convex closure of V

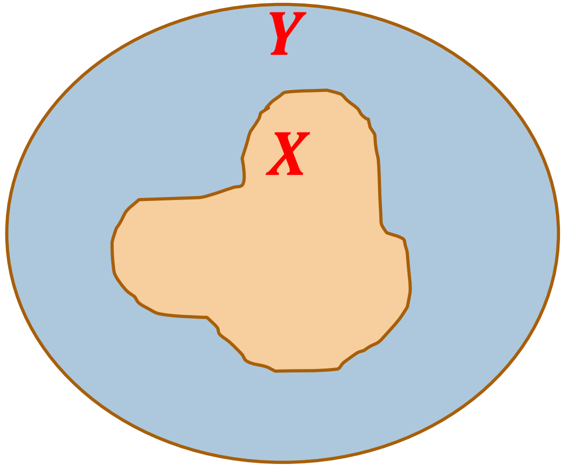


<u>Corollary</u>: If a problem is closed ($V = C(V)$) then oblivious rounding can achieve the integrality gap

Main Result Illustration

Problem (V, X) ⟷ Integrality gap ⟷ Relaxation (V, Y)

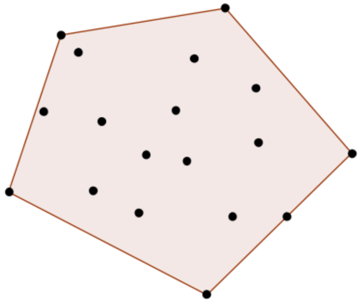
Oblivious rounding



$(C(V), X)$ Closure ⟷ Integrality gap of closure ⟷ $(C(V), Y)$ Relaxed closure

=

Oblivious rounding approximation ratio



Closure

An Application: Welfare Maximization

Informally: Allocate m indivisible items among buyers to maximize total value

- Each buyer i has a valuation function $v_i: 2^{[m]} \to \mathbb{R}_{\geq 0}$
- Valuations belong to classes (e.g., additive, submodular, …)

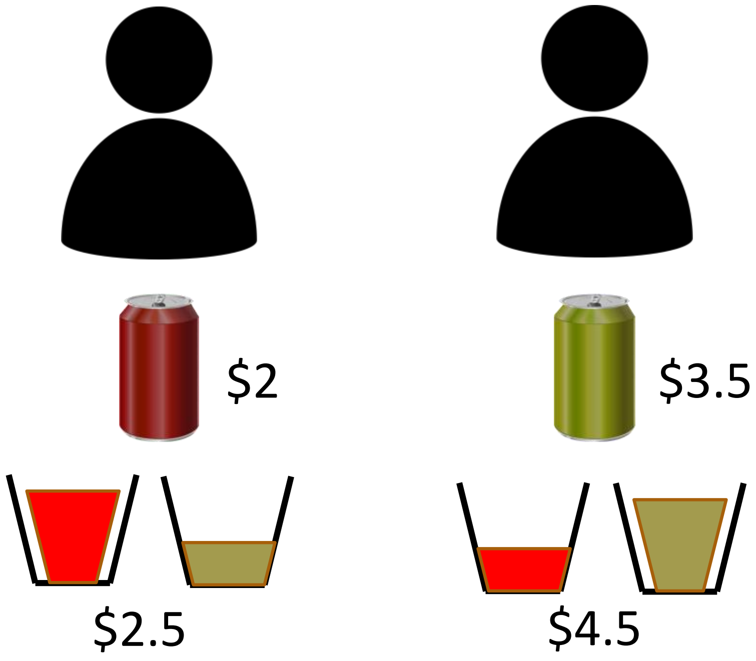
More formally:

- $v \in V$ = the buyers' valuations, from class V
- $x \in X$ = an allocation
- $y \in Y$ = an allocation as if the items were divisible

(all sets of vectors of dimension 2^m)

Main thm: Approximation ratio of oblivious rounding = integrality gap of problem's closure

Welfare Max: The Relaxation



Problem: Indivisible items — Welfare = 5.5

Relaxation: Divisible items — Welfare = $7

Integrality gap

Welfare Max: The Relaxation $X \to Y$

The relaxation used in ≈all welfare approximation algorithms: Configuration LP

Problem:

$\max_{i,S} \sum x_{i,S} v_{i,S}$

s.t.

$\sum_S x_{i,S} \leq 1$ for every buyer i

$\sum_{i,S:j\in S} x_{i,S} \leq 1$ for every item j

$x_{i,S} \in \{0,1\}$

Relaxation:

$\max_{i,S} \sum y_{i,S} v_{i,S}$

s.t.

$\sum_S y_{i,S} \leq 1$ for every buyer i

$\sum_{i,S:j\in S} y_{i,S} \leq 1$ for every item j

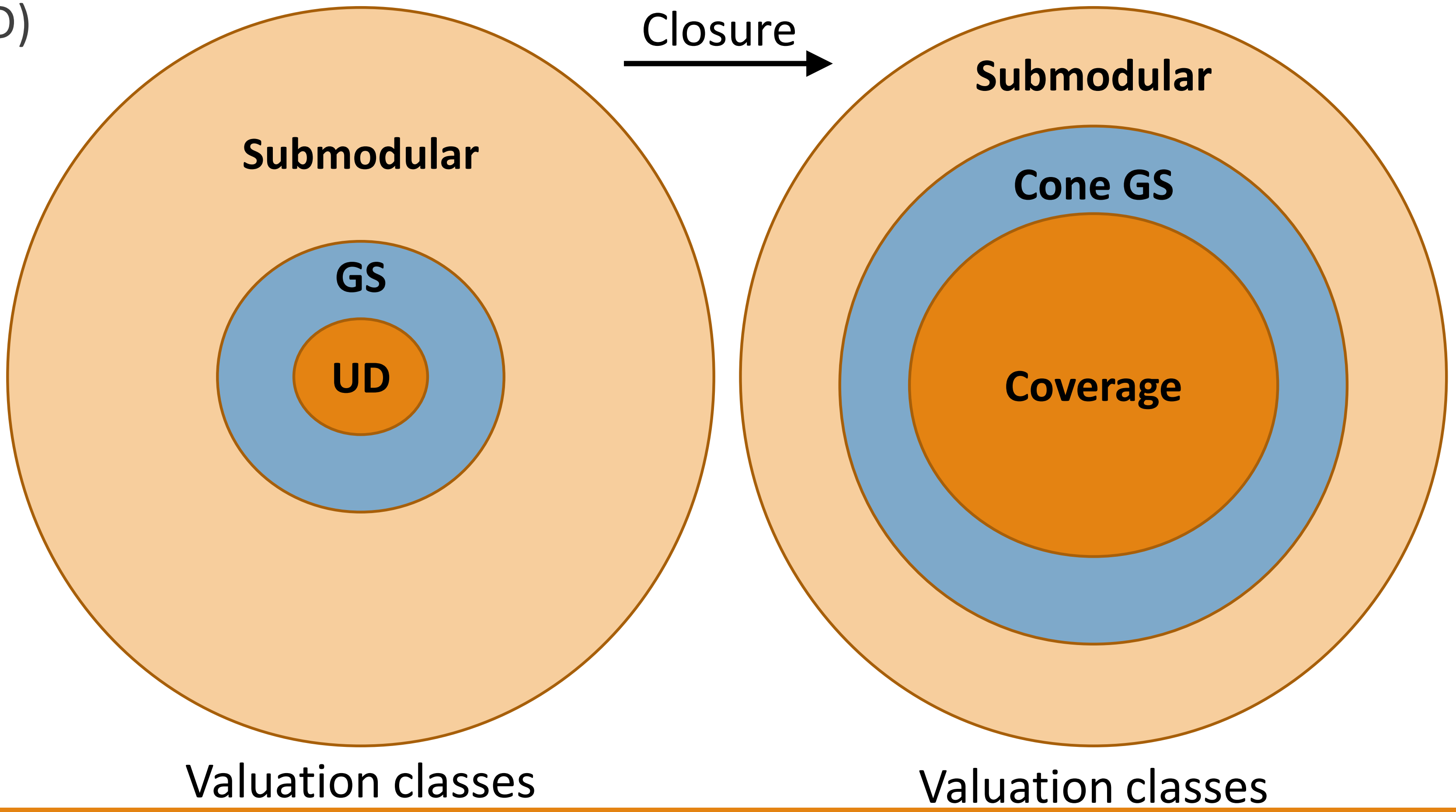
$y_{i,S} \geq 0$

Main thm: Approximation ratio of oblivious rounding = integrality gap of problem's closure

Welfare Max: The Closure $V \rightarrow C(V)$

Let V be unit-demand (UD) valuations:

- $v_i(S) = \max_{\text{item } j \in S} \{v(j)\}$
- Subclass of gross substitutes (GS)
- So integrality gap of configuration LP $= 1$



Main thm: Approximation ratio of oblivious rounding = integrality gap of problem's closure

Main Result Applied to Welfare Max.

Theorem: For welfare maximization with unit-demand valuations, oblivious rounding of solutions to the configuration LP achieves $\leq$ 0.782 approximation (0.833 for 2 buyers)

- Despite the integrality gap of 1 for unit-demand/GS

Conclusion: For welfare maximization, "ignorance is not always bliss"

- Need to know the valuations to round the fractional allocation

Proof: The integrality gap of the configuration LP for coverage valuations is no better than 0.782 [*cf.* Feige-Vondrak'10], and coverage is the closure of unit-demand

Implications for Mechanism Design

Advantages of oblivious rounding for welfare maximization with strategic buyers:

1. Incentive compatibility
   - [Duetting-Kesselheim-Tardos'15]: Can embed into a mechanism that approximately maximizes welfare in equilibrium
2. Fairness
   - Treats buyers equally, approximation ratio holds per buyer
3. Communication
   - Does not access exponential-sized valuations

Motivates understanding the possibilities/limitations of oblivious rounding

Sketch of Main Proof

Main thm: Approximation ratio of oblivious rounding = integrality gap of problem's closure

Approx. Ratio and Integrality Gap at y

Fix a fractional solution $y \in Y$

INTEGRALITY GAP OF CLOSURE AT y

$$\inf_{v \in C(V)} \max_{x \in X} \frac{v \cdot x}{v \cdot y}$$

First choose worst-case v from the closure

Then find the best integral solution x

APPROXIMATION RATIO OF BEST OBLIVIOUS ROUNDING AT y

$$\max_{x \in C(X)} \inf_{v \in V} \frac{v \cdot x}{v \cdot y}$$

First choose best randomized rounding

Then find the worst-case v for this rounding

This is where the obliviousness comes in

Main thm: Approximation ratio of oblivious rounding = integrality gap of problem's closure

Applying the Minimax Theorem

Fix a fractional solution $y \in Y$

MINIMIZING "OBJECTIVE" PLAYER

$$\inf_{v \in C(V)} \max_{x \in X} \frac{v \cdot x}{v \cdot y}$$

Choose minimizing mixed strategy

MAXIMIZING "ROUNDING" PLAYER

$$\max_{x \in C(X)} \inf_{v \in V} \frac{v \cdot x}{v \cdot y}$$

Choose maximizing mixed strategy

Summary

Many commonly-used rounding schemes are oblivious

- Do not use the objective to round a fractional solution

We study when oblivious rounding suffices for good approximation

- Focus on information

Approximation ratio equals the integrality gap of a related problem – the closure

Application to welfare maximization

- Oblivious rounding does not suffice for unit-demand, gross substitutes
- Suffices for submodular

Another tool for the toolbox of algorithm and mechanism designers

Directions for Future Research

1. Use the understanding of the potential and limitations of oblivious rounding as a guide in designing rounding schemes
   - for problems for which tight approximation ratios not yet known
   - e.g., when best known approximation is oblivious but the problem is not closed
2. Possibilities/limitations of polynomial-time oblivious rounding
3. Other properties of combinatorial problems predicting the success/failure of rounding techniques