The Competition Complexity of Auctions: Bulow-Klemperer Results for Multidimensional Bidders

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*Based on slides by Alon Eden

Complexity in AMD

One **goal** of **A**lgorithmic **M**echanism **D**esign:

Deal with complex allocation of goods settings

- Goods may **not** be **homogenous**
- **Valuations** and constraints may be **complex**

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- Goods may **not** be **homogenous**
- **Valuations** and constraints may be **complex**
- E.g. spectrum auctions, cloud computing, ad auctions, …

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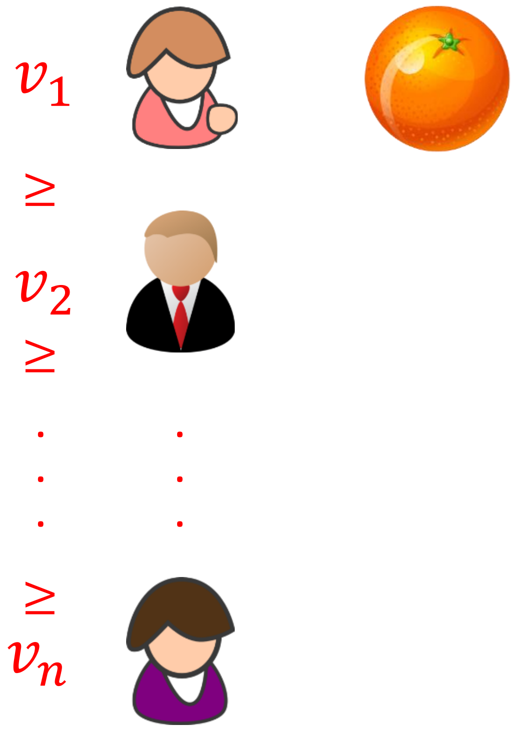
- **Resource augmentation**

Single item welfare maximization

Run a 2nd price auction – simple, maximizes welfare "**pointwise**".

(VCG mechanism)



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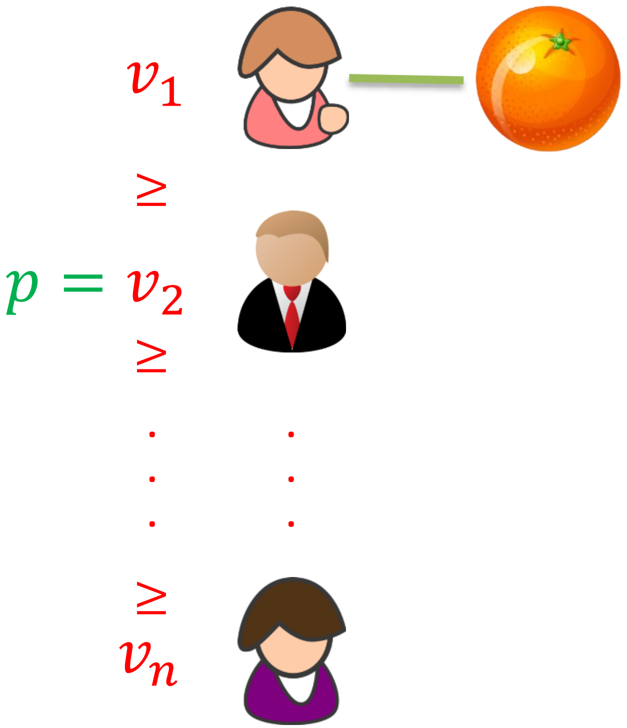

Single item revenue maximization

Single buyer: select price that maximizes $p \cdot (1 - F(p))$ ("monopoly price").

Price $= p$

$v_1 \sim F$

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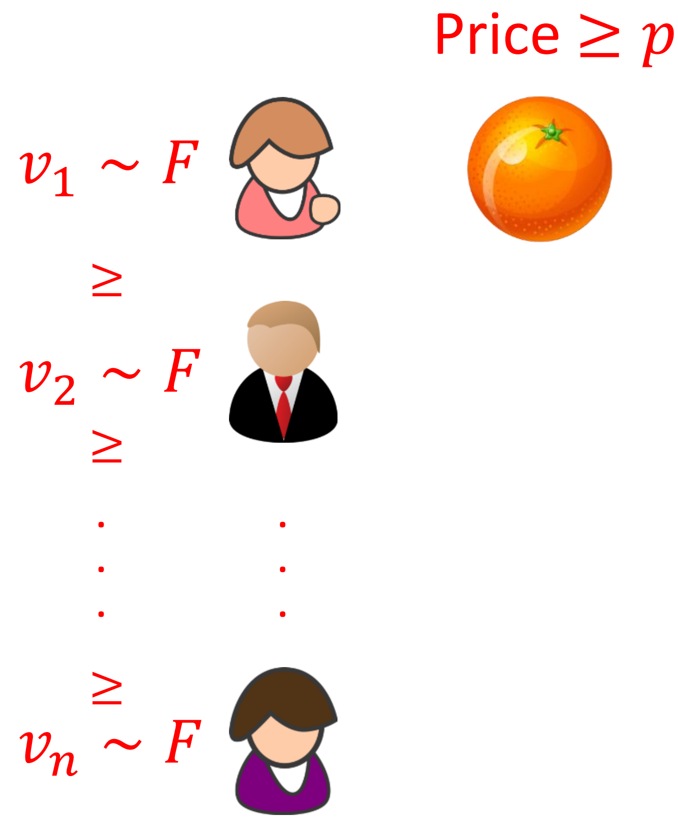
Multiple i.i.d. buyers: run 2nd price auction with reserve price p (same p).

(Myerson's auction)



Single item revenue maximization

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Multiple i.i.d. buy

2nd price auction v

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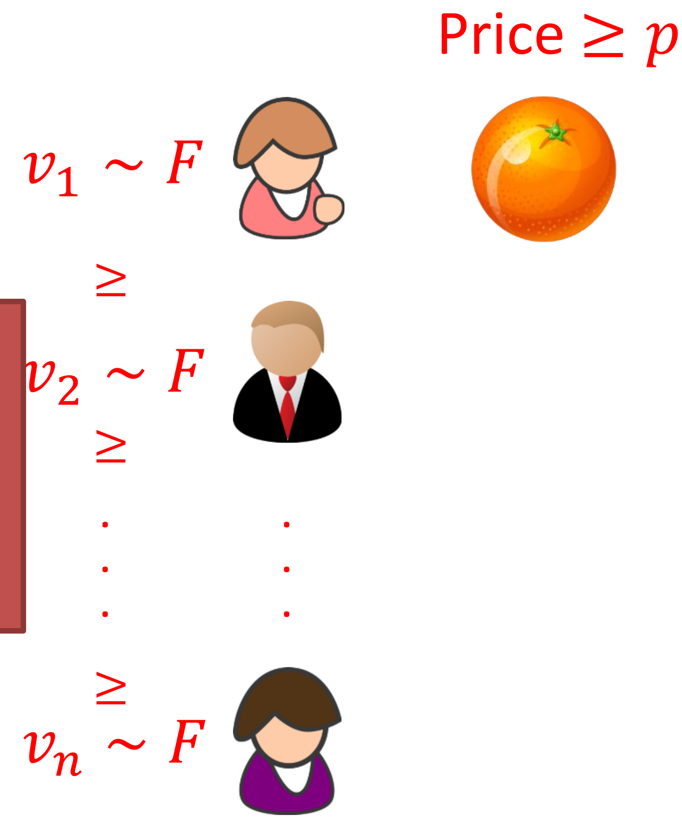
Assuming regularity

(Myerson's auction)



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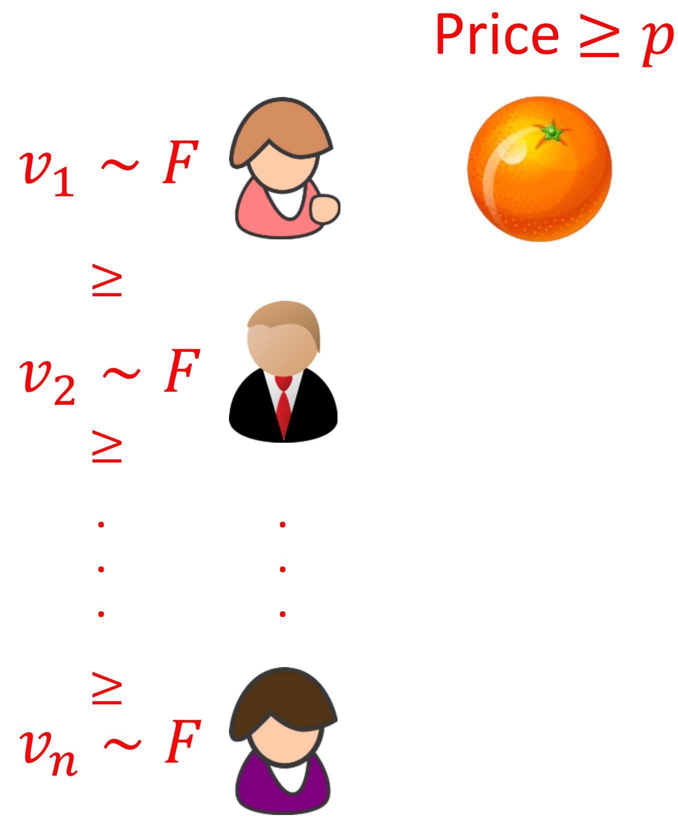

(Myerson's auction)

Requires prior knowledge to determine the reserve

Bulow-Klemperer theorem

Thm. Expected revenue of the **2nd price** auction with **n+1** bidders $\geq$ Expected revenue of the **optimal auction** with **n** bidders.

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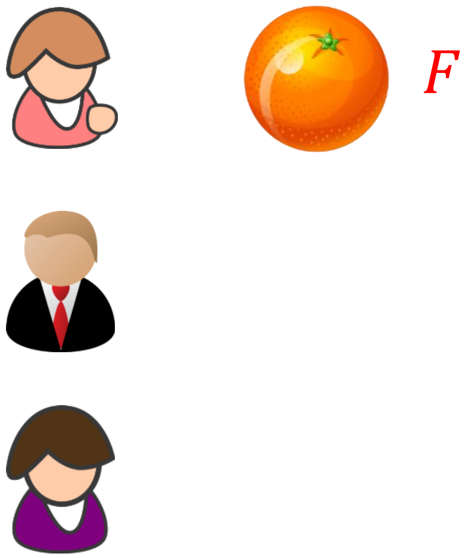
Robust! No need to learn the distribution. No need to change mechanism if the distribution changes. "The statistics of the data shifts rapidly" [Google]

Simple! "Hardly anything matters more" [Milgrom'04]

Multidimensional settings



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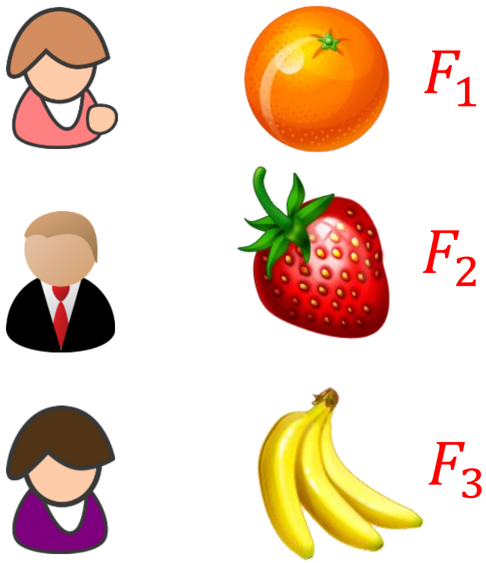
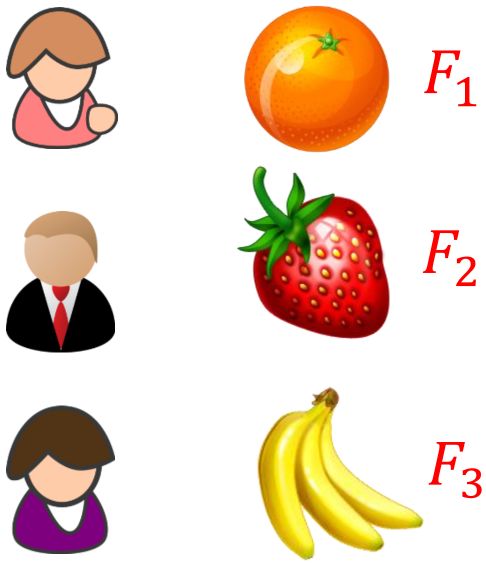


Multidimensional settings

Bidders' values are sampled i.i.d. from a product distribution over items



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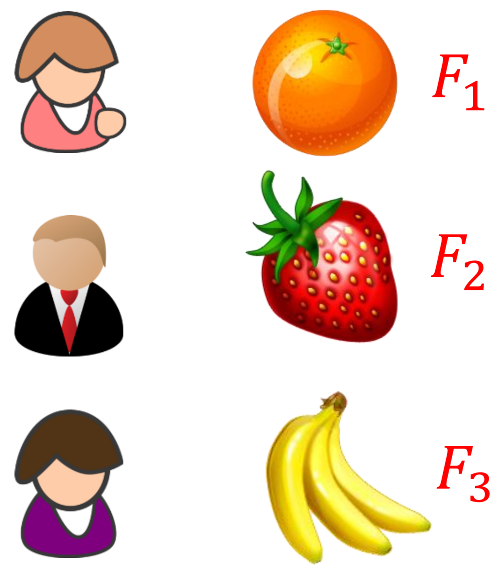
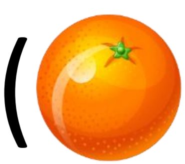
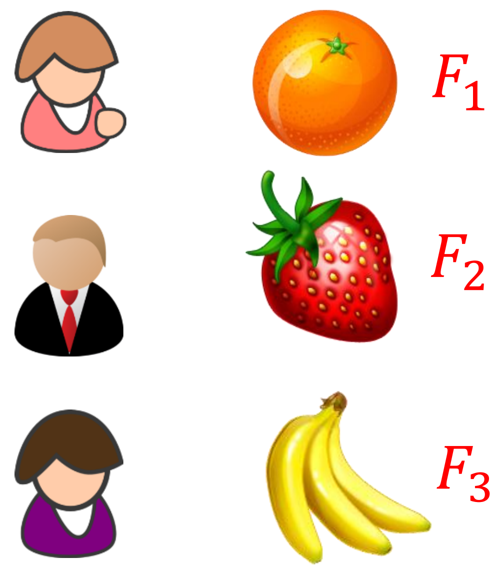


Additive: $v($,$)=v($$)+v($$)$

Multidimensional settings

- Revenue maximization is not well understood:
- Optimal mechanism might necessitate **randomization**.
- **Non-monotone**.
- Computationally **intractable**.
- Only recently, **simple approximately optimal** mechanisms were devised.



Multidimensional settings

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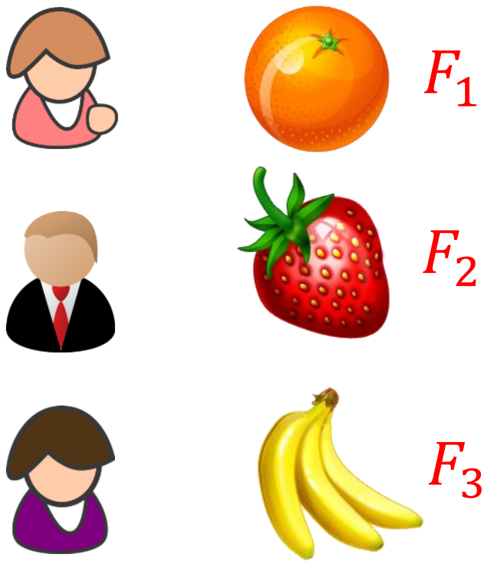

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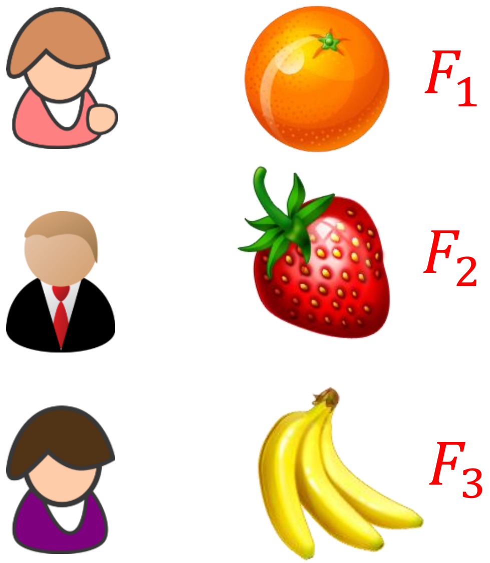

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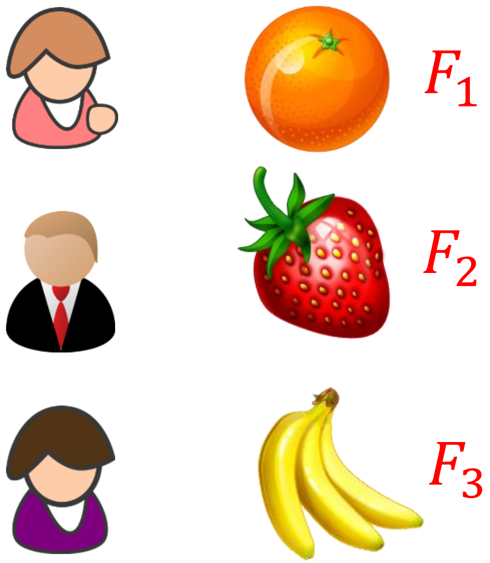

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Either run a **randomized**,
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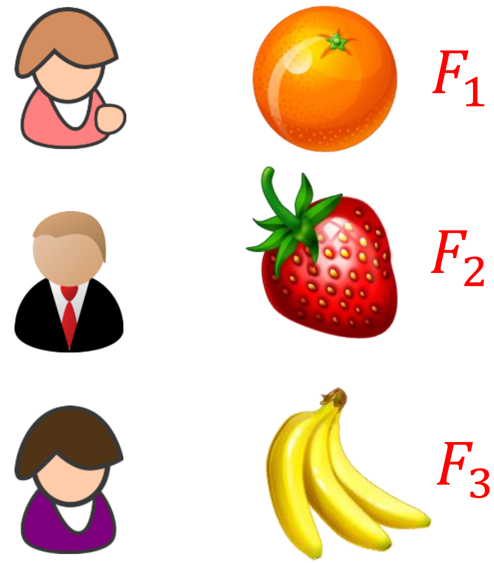

Multidimensional settings

Either run a **randomized**,
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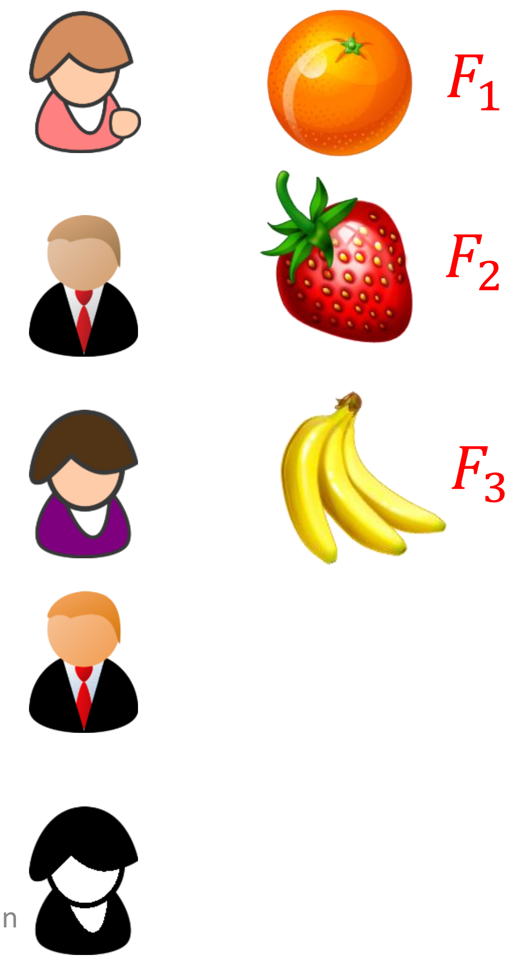
Or add **more bidders**.



OUR RESULTS

Multidimensional B-K theorems

Competition complexity: Fix an environment with n i.i.d. bidders. **What is x** such that the revenue of **VCG** with $n + x$ bidders is $\geq$ **OPT** with n bidders.

Multidimensional B-K theorems

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Bulow-Klemperer Thm. The competition complexity of a **single item** auction is **1**.

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Thm. [BK] The competition complexity of a single item **with m copies** is m.

Multidimensional B-K theorems

Competition complexity: Fix an environment with n i.i.d. bidders. **What is x** such that the revenue of **VCG** with $n + x$ bidders is $\geq$ **OPT** with n bidders.

Thm. [EFFTW] The competition complexity of n **additive bidders** drawn from a product distribution over m **items** is $\leq n + 2(m-1)$.

Multidimensional B-K theorems

Competition complexity: Fix an environment with n i.i.d. bidders. **What is x** such that the revenue of **VCG** with $\boldsymbol{n+x}$ bidders is $\geq$ **OPT** with $\boldsymbol{n}$ bidders.

Thm. [EFFTW] Let C be the competition complexity of $\boldsymbol{n}$ **additive bidders** over m items. The competition complexity of n **additive bidders** with **identical downward closed constraints** over m items is $\leq \boldsymbol{C+m-1}$.

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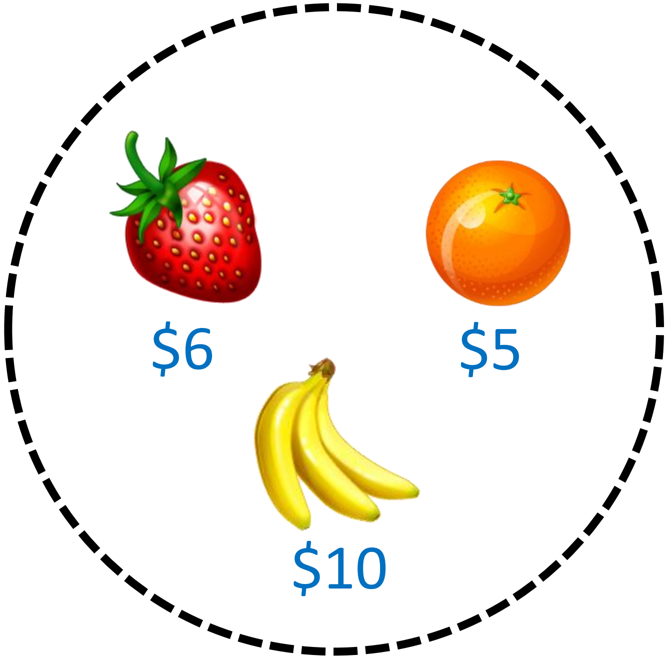
Additive with constraints

- **Constraints** = set system over the items
  - Specifies which item sets are **feasible**
- Bidder's value for an item set = her value for **best feasible subset**
- If all sets are feasible, bidder is **additive**

Example of constraints

- No constraints

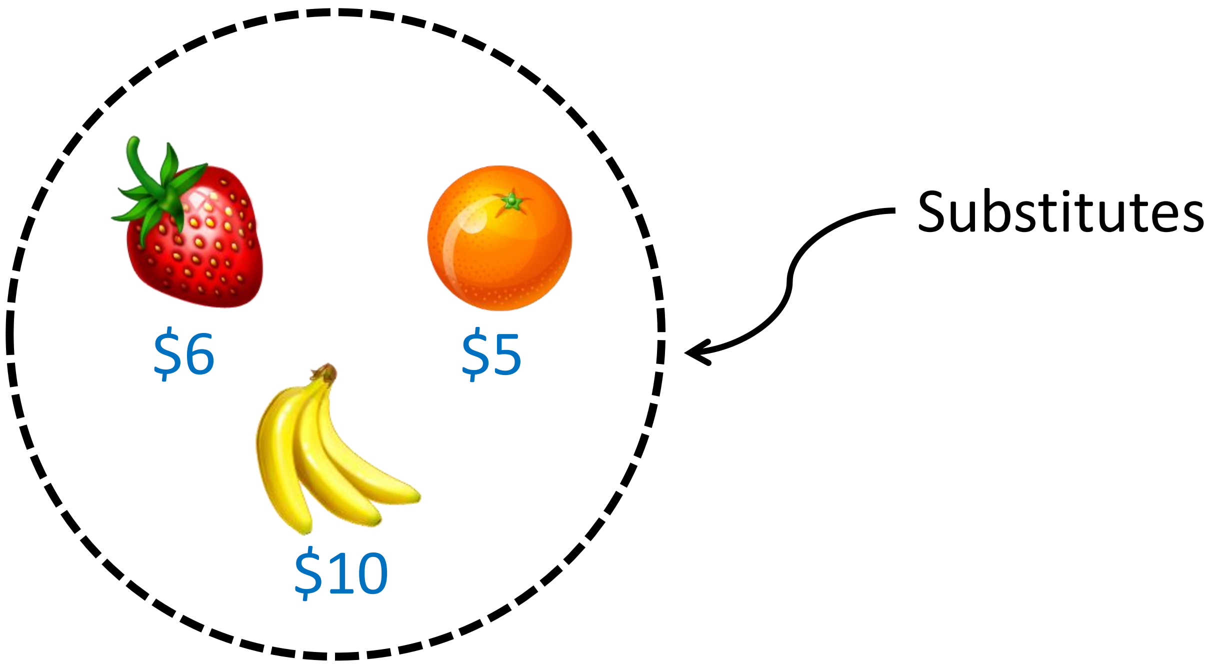
Total value = $21



Example of constraints

- Example of "**matroid**" constraints: Only sets of size $k = 1$ are feasible

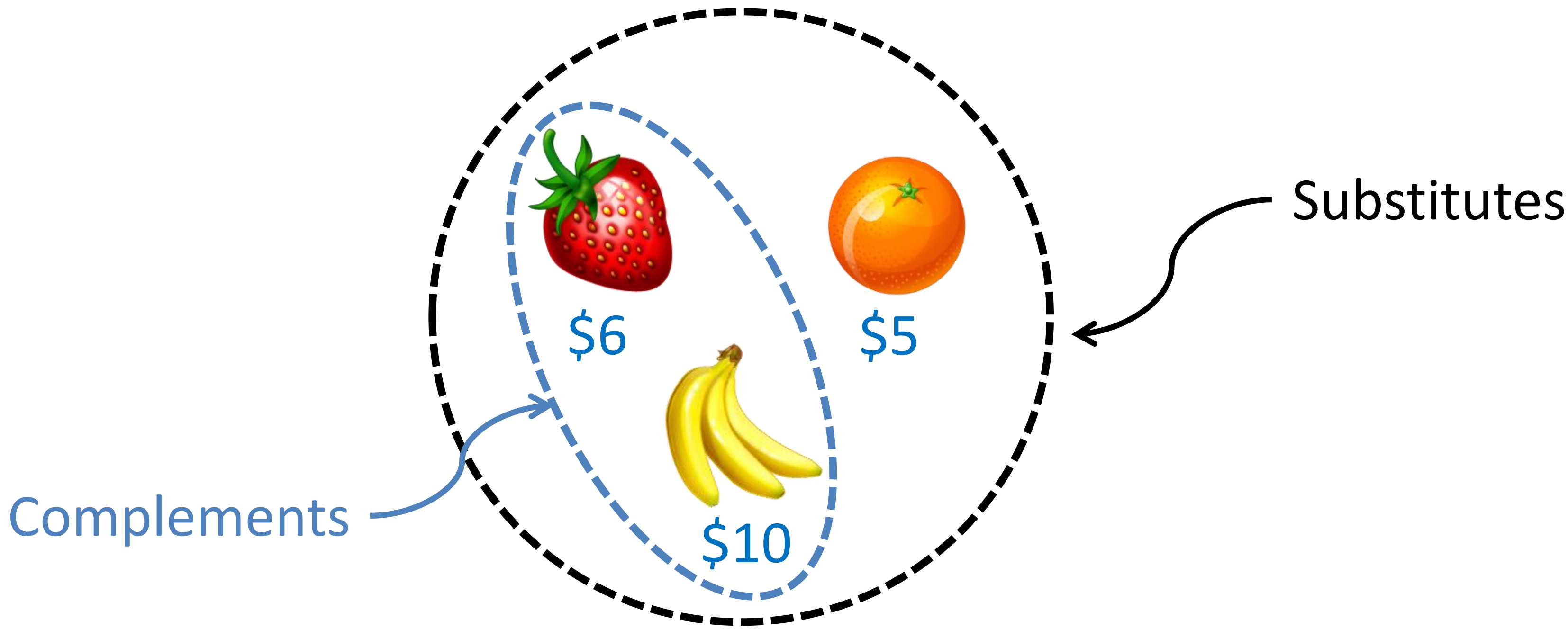
Total value = $10



Example of constraints

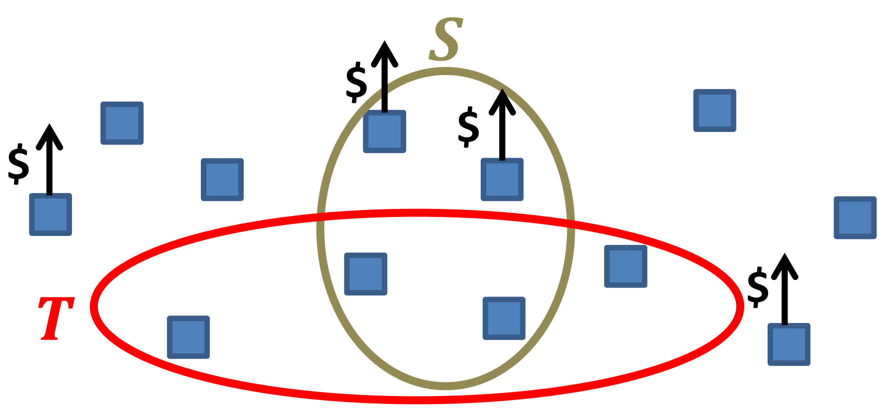
- Example of "**downward closed**" constraints:
  Sets of size 1 and {} are feasible

Total value = $10



Complements in what sense?

- No complements = gross substitutes:
  - $\vec{p} \leq \vec{q}$ item prices
  - S in demand($\vec{p}$) if maximizes utility $v_i(S) - p(S)$
  - $\forall S$ in demand($\vec{p}$), there is T in demand($\vec{q}$) with every item in S **whose price didn't increase**



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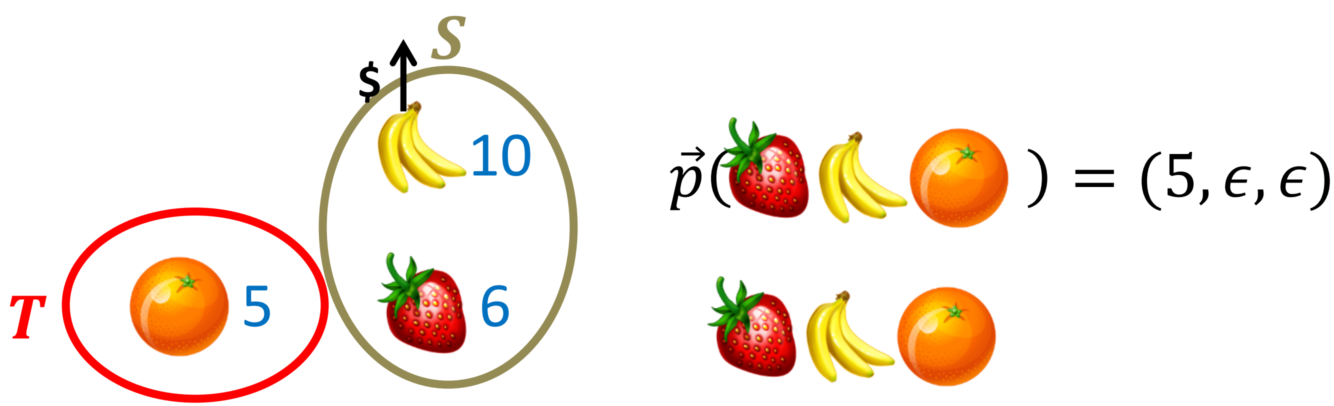

Competition complexity – summary

Valuation	Upper bound
Additive	$n + 2(m-1)$
Additive s.t. **identical** downward closed constraints	$n + 3(m-1)$
Additive s.t. **random** downward closed constraints	$n + 4(m-1)$
Additive s.t. identical **matroid** constraints	$n + 2(m-1) + \rho$

Lower bounds of $\Omega\left(n \cdot \log\left(\frac{n}{m} + 1\right)\right)$ for additive bidders and $\Omega(m)$ for unit demand bidders are due to ongoing work by **[Feldman-Friedler-Rubinstein]** and to **[Bulow-Klemperer'96]**

Related work

Multidimensional B-K theorems

[Roughgarden T. Yan '12]: for **unit demand** bidders, revenue of **VCG** with m **extra bidders** $\geq$ revenue of the optimal **deterministic DSIC** mechanism.

[Feldman Friedler Rubinstein – ongoing]: tradeoffs between enhanced competition and revenue.

Prior-independent multidimensional mechanisms

[Devanur Hartline Karlin Nguyen '11]: **unit demand** bidders.

[Roughgarden T. Yan '12]: **unit demand** bidders.

[Goldner Karlin '16]: **additive** bidders.

Sample complexity

[Morgenstern Roughgarden '16]: **how many samples** needed to **approximate** the **optimal** mechanism?

MULTIDIMENSIONAL B-K THEOREM PROOF SKETCH

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III. Show that the 2nd price auction "beats" A.

Proof:

Myerson's optimal mechanism

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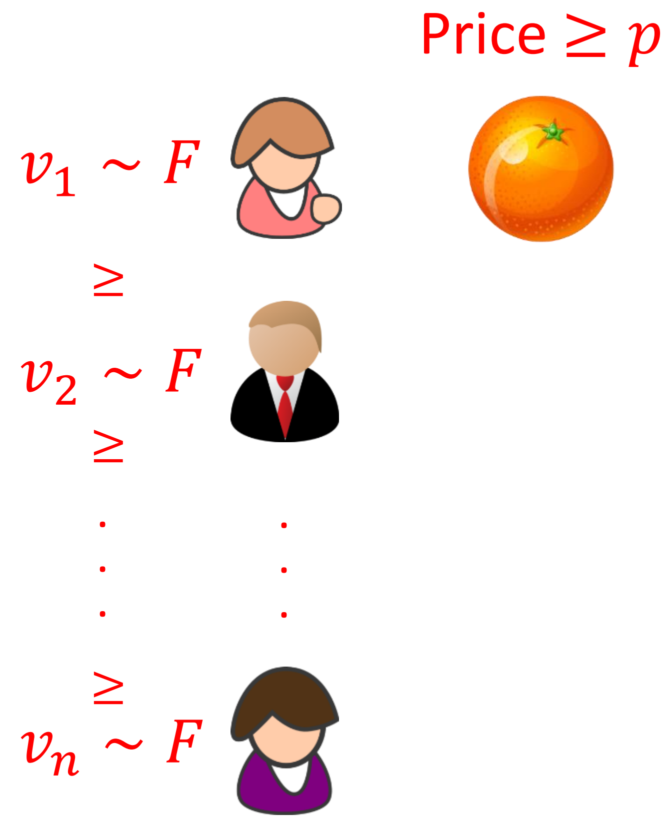
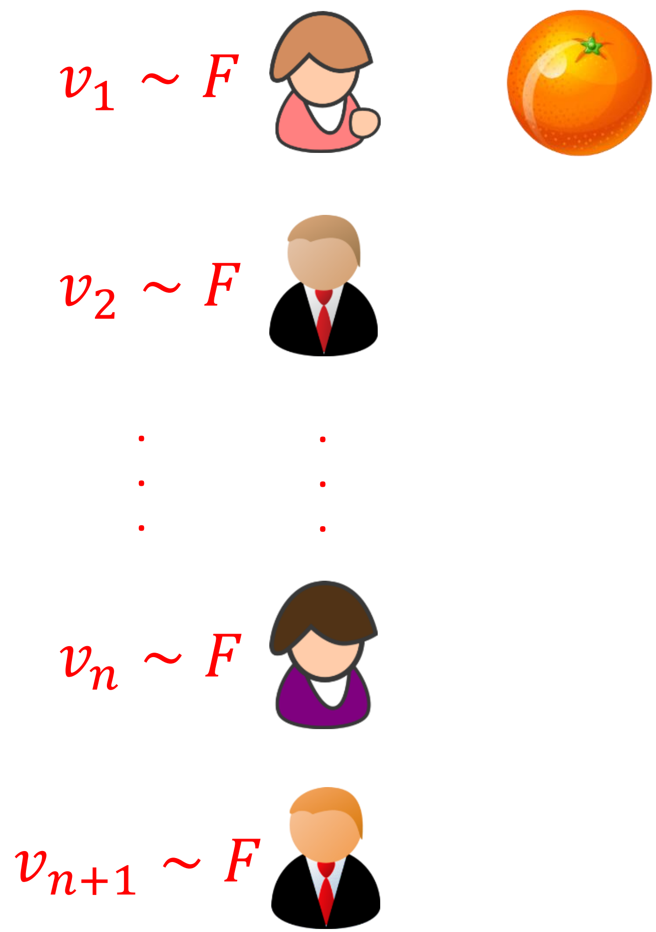


Proof:

Step II. Find an auction A with more bidders and revenue $\geq$ the upper bound.



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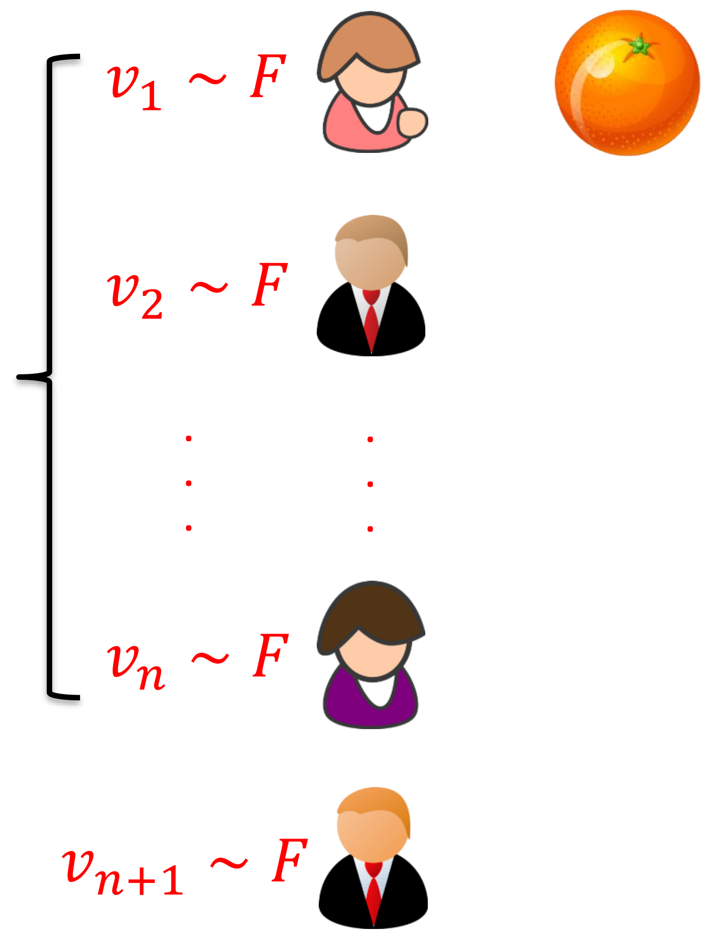
Run Myerson's mechanism on n bidders



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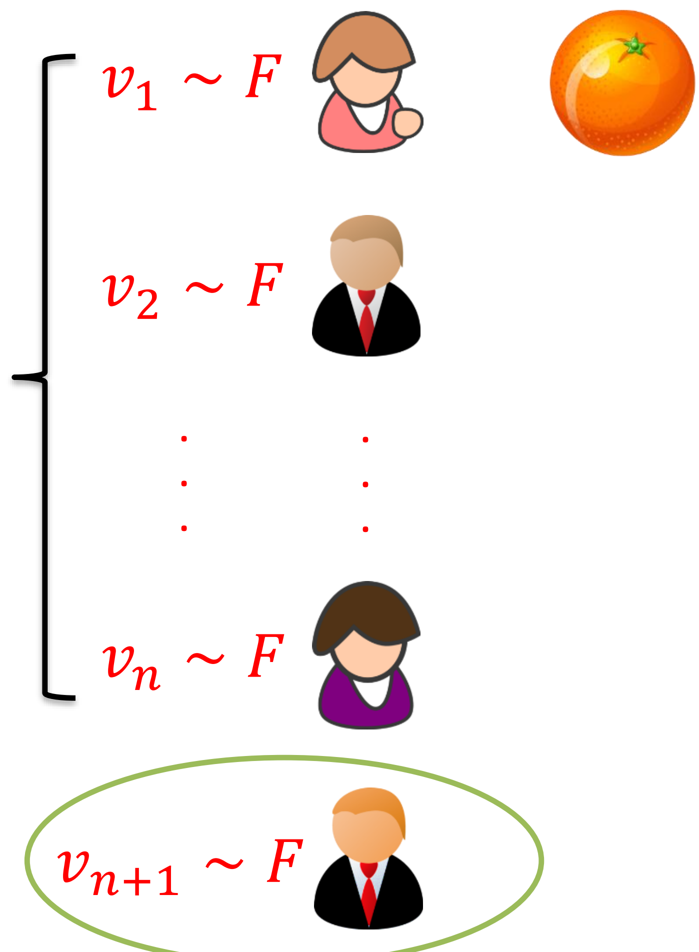
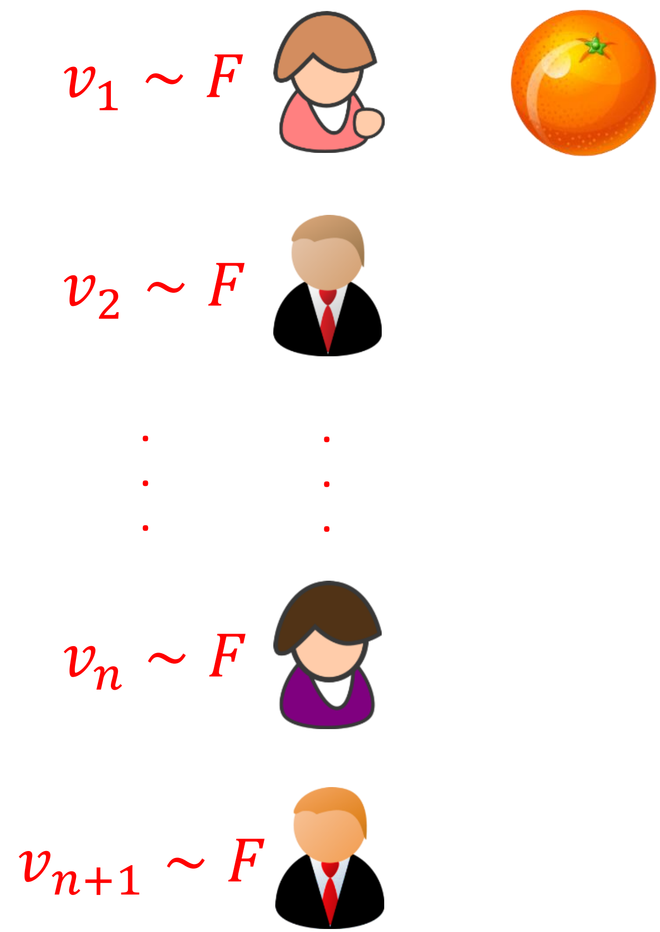


If Myerson does not allocate, give item to the additional bidder

Proof:

Step III. Show that the 2nd price auction "beats" A.

Observation. 2nd price auction is the optimal mechanism out of the mechanisms that always sell.



Competition complexity of a single additive bidder

Plan: Follow the 3 steps of the B-K proof.

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Competition complexity of a single additive bidder and i.i.d. items

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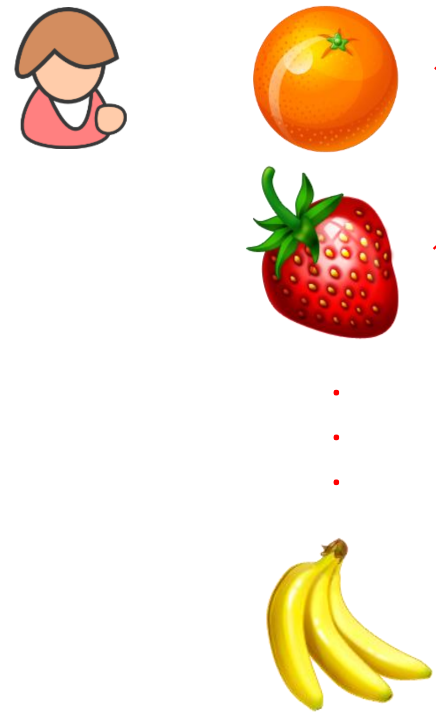
I. Upper-bound the optimal revenue

- Single additive bidder and i.i.d. items



$v_1 \sim F$

$v_2 \sim F$

$\vdots \quad \vdots$

$v_m \sim F$

I. Upper-bound the optimal revenue

Use the duality framework from **[Cai Devanur Weinberg '16]**.

$$\mathrm{OPT} \leq \mathrm{E}_{v\sim F^m}\left[\sum_j \varphi^+(v_j)\cdot 1_{\forall j'\, v_j > v_{j'}} + v_j \cdot 1_{\exists j'\, v_j < v_{j'}}\right]$$

$\varphi(v) = v - \frac{1-F(v)}{f(v)}$ is the virtual valuation function.

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Distribution appears in proof only!

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Take item j's virtual value if it's the most attractive item

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II. Find an auction A with m bidders and revenue $\geq$ upper bound

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VCG for **additive bidders** $\equiv$ **2nd price auction** for **each item** separately.

Therefore, we devise a **single parameter mechanism** that covers item ***j*'s contribution** to the benchmark.

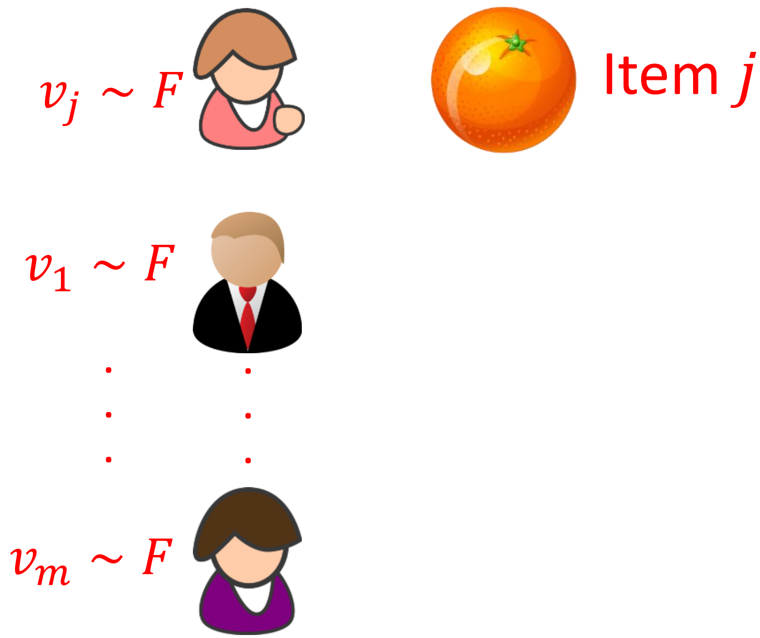
$$\mathrm{E}_{v\sim F^m}\left[\varphi^+(v_j)\cdot 1_{\forall j'\ v_j > v_{j'}} + v_j \cdot 1_{\exists j'\ v_j < v_{j'}}\right]$$

II. Find an auction $A(j)$ with m bidders and revenue ≥ upper bound for item j

$$\mathrm{E}_{v\sim F^m}\left[\varphi^+(v_j)\cdot 1_{\forall j'\ v_j>v_{j'}} + v_j \cdot 1_{\exists j'\ v_j<v_{j'}}\right]$$

Run 2nd price auction with "lazy" reserve price =

$$\begin{cases} \varphi^{-1}(0) \text{ for agent } j \\ 0 \text{ for agents } j' \neq j \end{cases}$$



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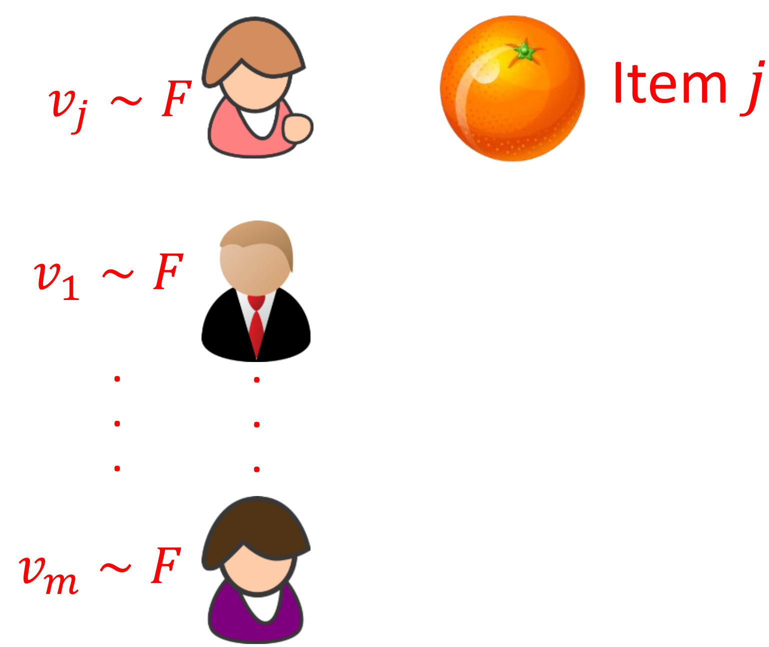
$$\mathrm{E}_{v\sim F^m}\left[\varphi^+(v_j)\cdot 1_{\forall j'\, v_j>v_{j'}} + v_j \cdot 1_{\exists j'\, v_j<v_{j'}}\right]$$

Case I: $v_j > v_{j'}$ for all j':

j wins if his virtual value is non-negative.

Expected revenue = Expected virtual value [Myerson'81]



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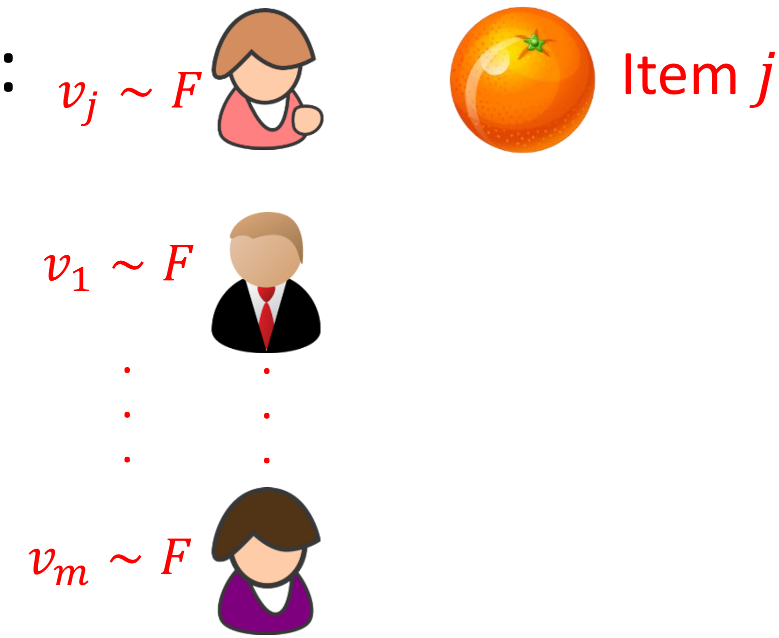
Case II: $v_j < v_{j'}$ for some j':

The second price is at least the value of agent j.



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$\boldsymbol{A(j)}$ with $\boldsymbol{m}$ bidders $\leq$ **Myerson** with $\boldsymbol{m}$ bidders $\leq$ **2nd price** with $\boldsymbol{m+1}$ bidders

III. Show that 2nd price "beats" $A(j)$

$A(j)$ with m bidders	$\leq$	**Myerson** with m bidders	$\leq$	**2nd price** with $m+1$ bidders

→ The competition complexity of a single additive bidder and m i.i.d. items is $\leq m$.

FF

Going beyond i.i.d items

- Single additive bidder and i.i.d. items

$v_1 \sim F_1$

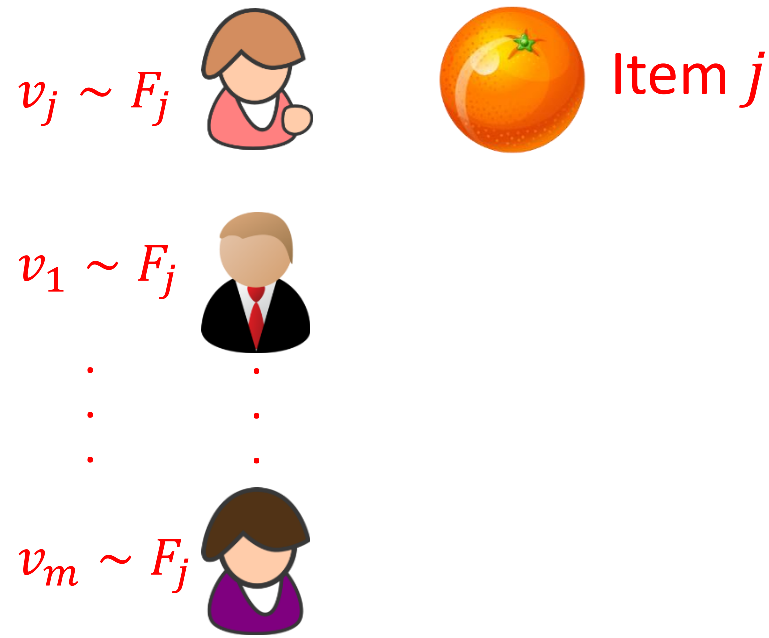
$v_2 \sim F_2$

$\vdots \quad \vdots$

$v_m \sim F_m$

Going beyond i.i.d items

$$\mathrm{E}_{\substack{v_1 \sim F_1 \\ v_2 \sim F_2 \\ \dots \\ v_m \sim F_m}} \left[\varphi_j^+(v_j) \cdot 1_{\forall j' \, v_j > v_{j'}} + v_j \cdot 1_{\exists j' \, v_j < v_{j'}} \right]$$



Run 2nd price auction

with "lazy" reserve price =
$\varphi^{-1}(0)$ for agent j

0 for agents $j' \neq j$

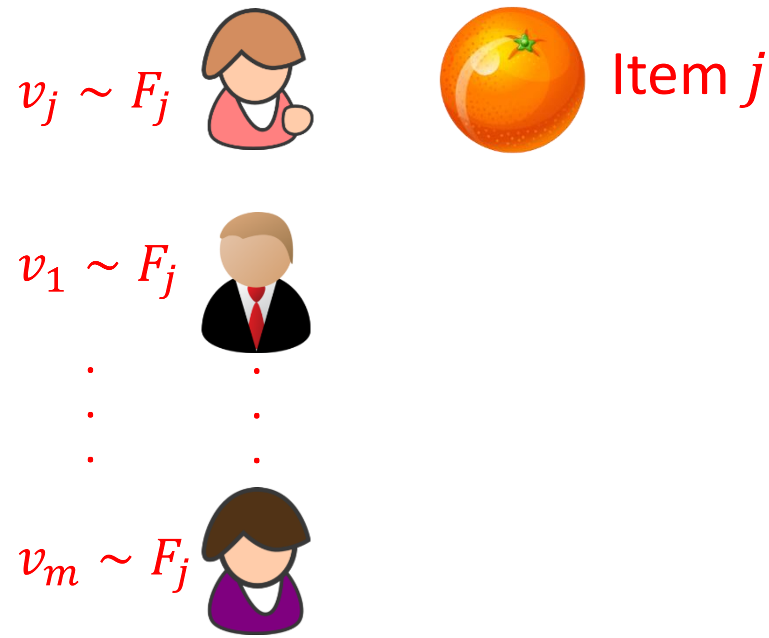
Going beyond i.i.d items

$$\mathrm{E}_{\substack{v_1 \sim F_1 \\ v_2 \sim F_2 \\ \cdots \\ v_m \sim F_m}} \left[\varphi_j^+(v_j) \cdot 1_{\forall j' \, v_j > v_{j'}} + v_j \cdot 1_{\exists j' \, v_j < v_{j'}} \right]$$

Run 2nd price auction
with "lazy" reserve price =

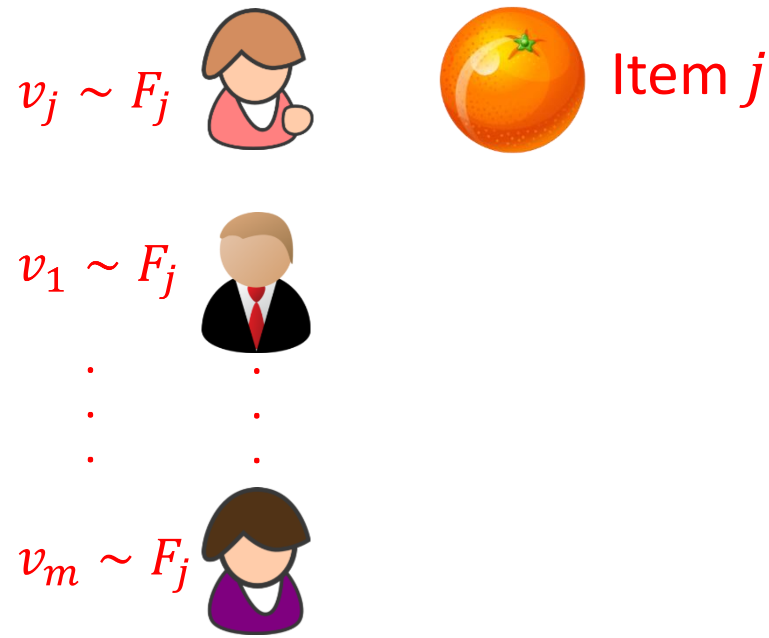
$$\begin{cases} \varphi^{-1}(0) \text{ for agent } j \\ 0 \text{ for agents } j' \neq j \end{cases}$$

Cannot couple the event
"bidder j wins" and
"item j has the highest value"



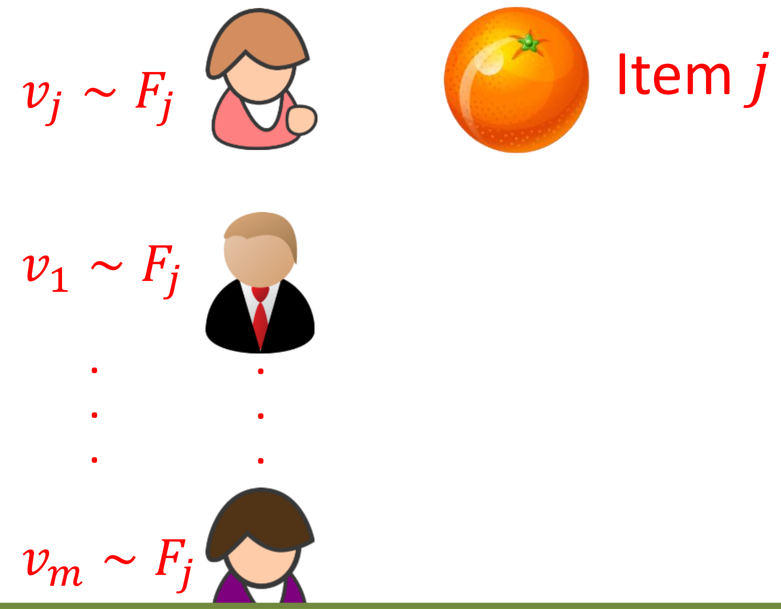
Use a different benchmark

$$\mathrm{E}_{\substack{v_1 \sim F_1 \\ v_2 \sim F_2 \\ \cdots \\ v_m \sim F_m}} \left[\varphi_j^+(v_j) \cdot 1_{\forall j' \; F_j(v_j) > F_{j'}(v_{j'})} + v_j \cdot 1_{\exists j' \; F_j(v_j) < F_{j'}(v_{j'})} \right]$$



Use a different benchmark

$$\mathrm{E}_{\substack{v_1\sim F_1\\ v_2\sim F_2\\ \dots\\ v_m\sim F_m}}\left[\varphi_j^+(v_j)\cdot 1_{\forall j'\ F_j(v_j)>F_{j'}(v_{j'})} + v_j\cdot 1_{\exists j'\ F_j(v_j)<F_{j'}(v_{j'})}\right]$$



→ The competition complexity of a single additive bidder and m items is $\leq m$.

Going beyond a single bidder

- Step I:
  - Benchmark more involved
- Step II:
  - Devise a more complex single parameter auction A(j) (involves a max)
  - Proving A(j) is greater than item j's contribution to the benchmark is more involved and requires subtle **coupling** and **probabilistic claims**

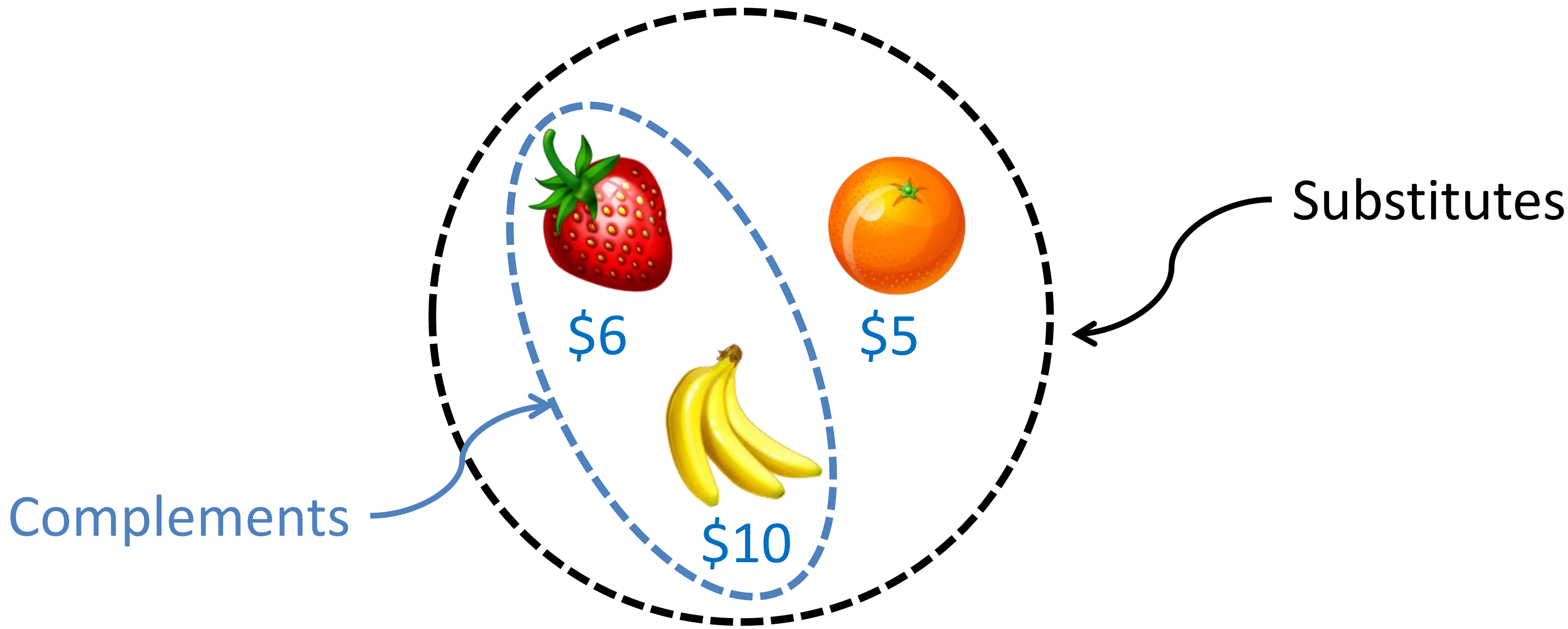
BB

EXTENSION TO CONSTRAINTS

Recall

- Example of "**downward closed**" constraints: Sets of size 1 and {} are feasible

Total value = $16



Extension to downward closed constraints

$$\mathrm{OPT}_n^{\mathrm{Add}} \leq \mathrm{VCG}_{n+C}^{\mathrm{Add}}$$

Competition complexity $\leq C$

Extension to downward closed constraints

$$\mathrm{OPT}_n^{\mathrm{DC}} \leq \mathrm{OPT}_n^{\mathrm{Add}} \leq \mathrm{VCG}_{n+C}^{\mathrm{Add}}$$

Larger outcome space

Competition complexity $\leq C$

Extension to downward closed constraints

$$\mathrm{OPT}_n^{\mathrm{DC}} \leq \mathrm{OPT}_n^{\mathrm{Add}} \leq \mathrm{VCG}_{n+C}^{\mathrm{Add}} \leq \mathrm{VCG}_{n+C+m-1}^{\mathrm{DC}}$$

Larger outcome space

Competition complexity $\leq C$

Extension to downward closed constraints

$$\mathrm{OPT}_n^{\mathrm{DC}} \leq \mathrm{OPT}_n^{\mathrm{Add}} \leq \mathrm{VCG}_{n+C}^{\mathrm{Add}} \leq \mathrm{VCG}_{n+C+m-1}^{\mathrm{DC}}$$

Larger outcome space

Competition complexity $\leq C$

The competition complexity of n additive bidders with identical downward closed constraints over m items is $\leq C + m - 1$.

Extension to downward closed constraints

Main technical challenge

$$\mathrm{OPT}_n^{\mathrm{DC}} \leq \mathrm{OPT}_n^{\mathrm{Add}} \leq \mathrm{VCG}_{n+C}^{\mathrm{Add}} \leq \mathrm{VCG}_{n+C+m-1}^{\mathrm{DC}}$$

Larger outcome space

Competition complexity $\leq C$

The competition complexity of n additive bidders with identical downward closed constraints over m items is $\leq C + m - 1$.

Claim. VCG revenue from selling **m items** to $X = n + C$ **additive bidders** whose values are i.i.d. draws from F

$$\leq$$

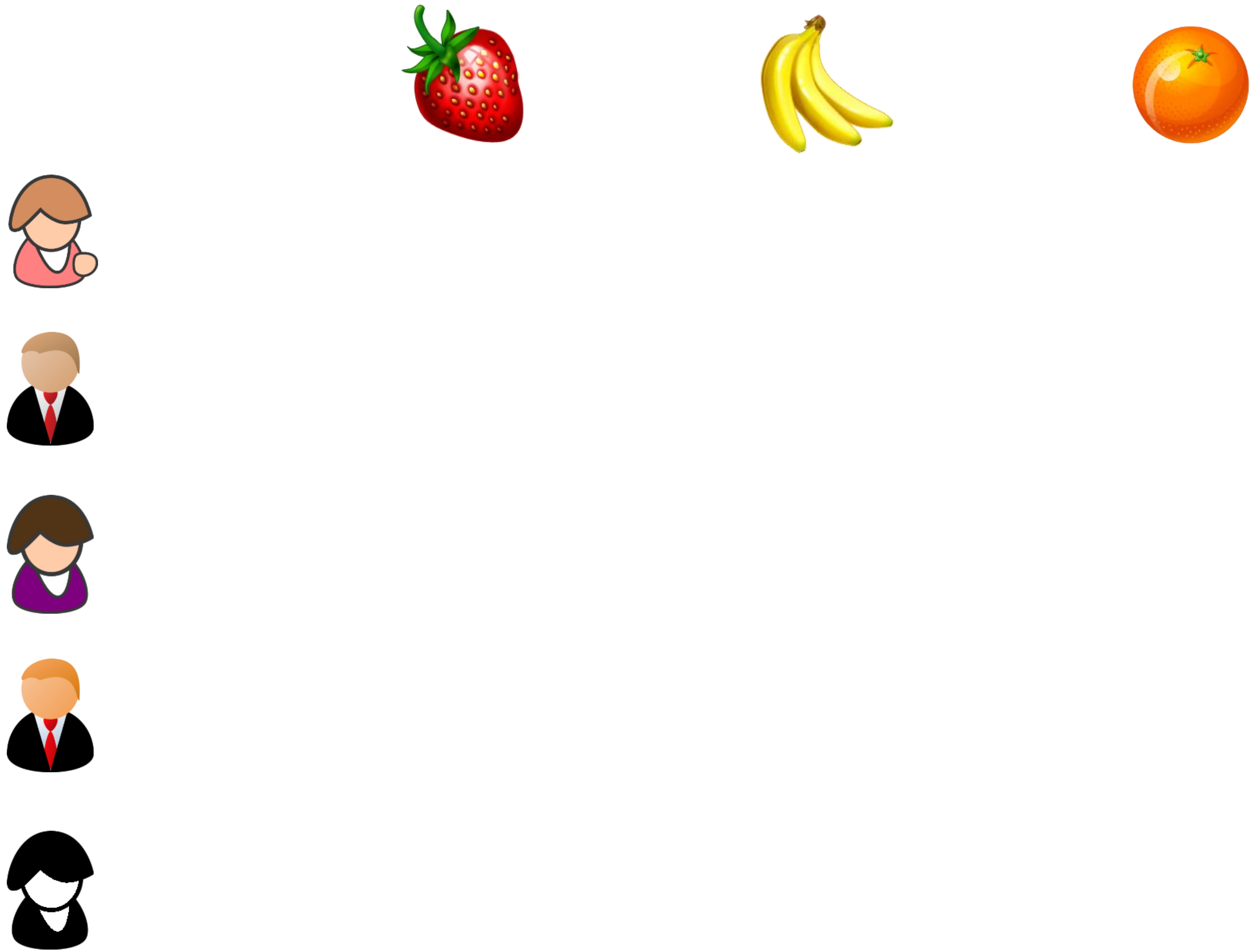
VCG revenue from selling them to **$X + m - 1$ bidders** with i.i.d. values drawn from F, whose valuations are **additive s.t. identical downward-closed** constraints.

$$\mathrm{VCG}_X^{\mathrm{Add}} \leq \mathrm{VCG}_{X+m-1}^{\mathrm{DC}}$$

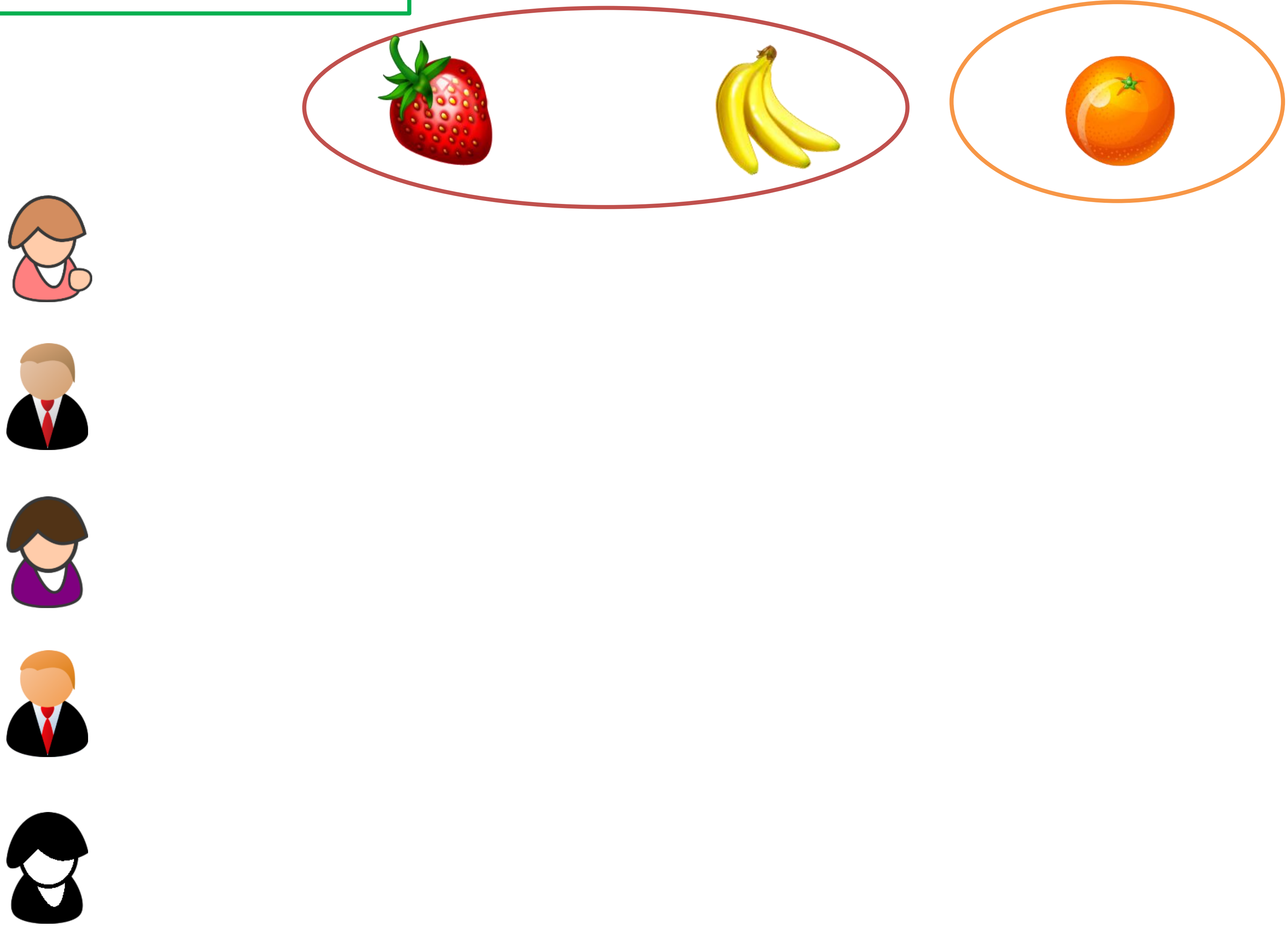
VCG for **additive bidders** ≡ **2nd price auction** for **each item** separately.

Therefore, the **revenue from item *j*** in $\mathrm{VCG}_X^{\mathrm{Add}}$ = **2nd highest** value out of X i.i.d. samples from F_j.

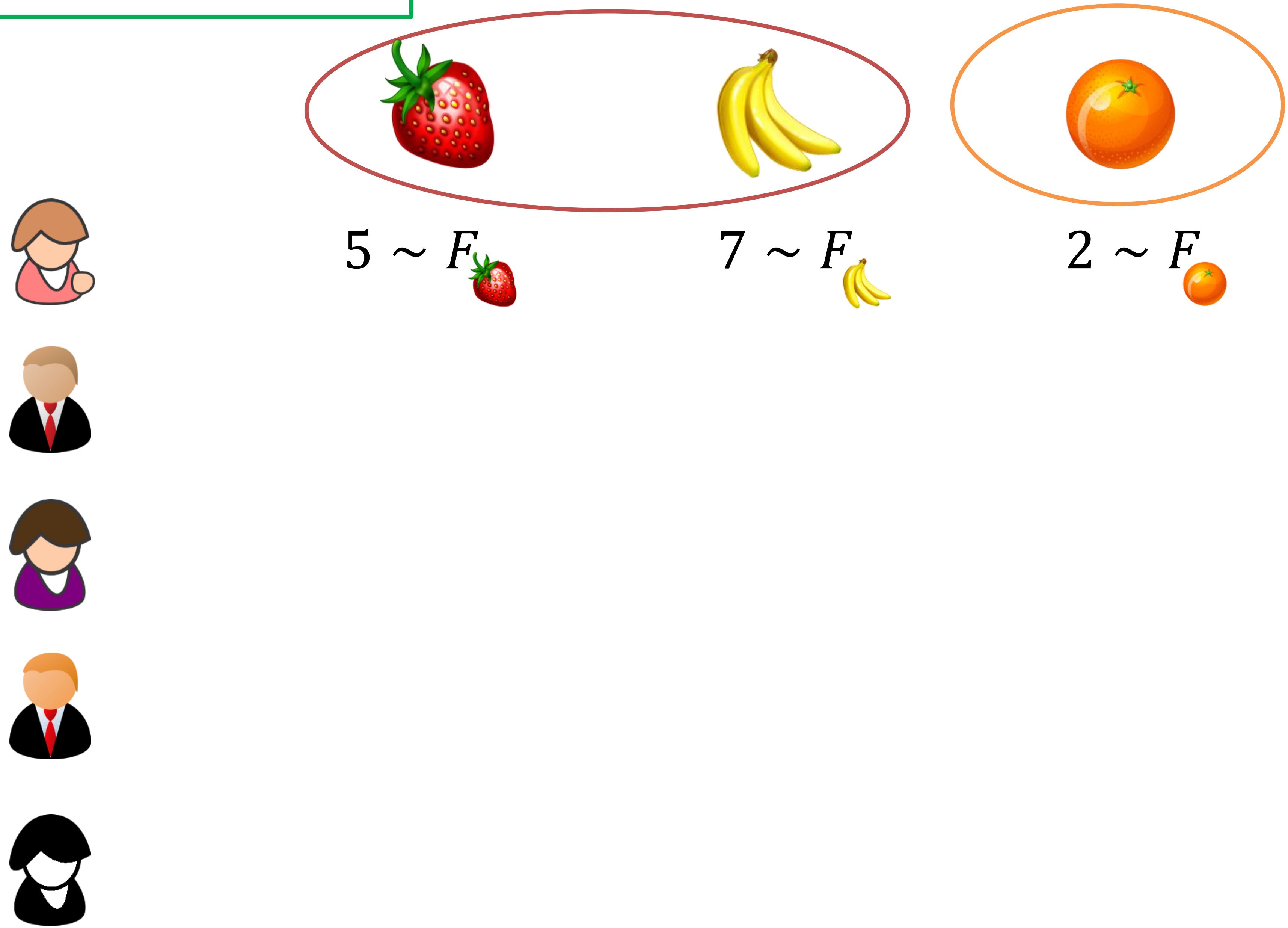
$$\mathrm{VCG}_{X}^{\mathrm{Add}} \leq \mathrm{VCG}_{X+m-1}^{\mathrm{DC}}$$



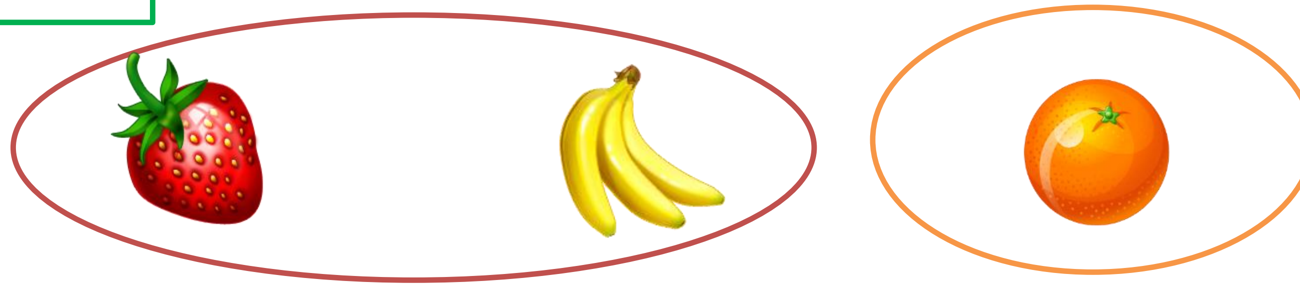
$$\mathrm{VCG}_X^{\mathrm{Add}} \leq \mathrm{VCG}_{X+m-1}^{\mathrm{DC}}$$



$$\mathrm{VCG}_X^{\mathrm{Add}} \le \mathrm{VCG}_{X+m-1}^{\mathrm{DC}}$$



$$\mathrm{VCG}_X^{\mathrm{Add}} \leq \mathrm{VCG}_{X+m-1}^{\mathrm{DC}}$$



$5 \sim F$	$7 \sim F$	$2 \sim F$
3	6	4
4	5	1
3	4	2
3	2	1

$$\mathrm{VCG}_X^{\mathrm{Add}} \leq \mathrm{VCG}_{X+m-1}^{\mathrm{DC}}$$

$5 \sim F$	$7 \sim F$	$2 \sim F$
3	6	4
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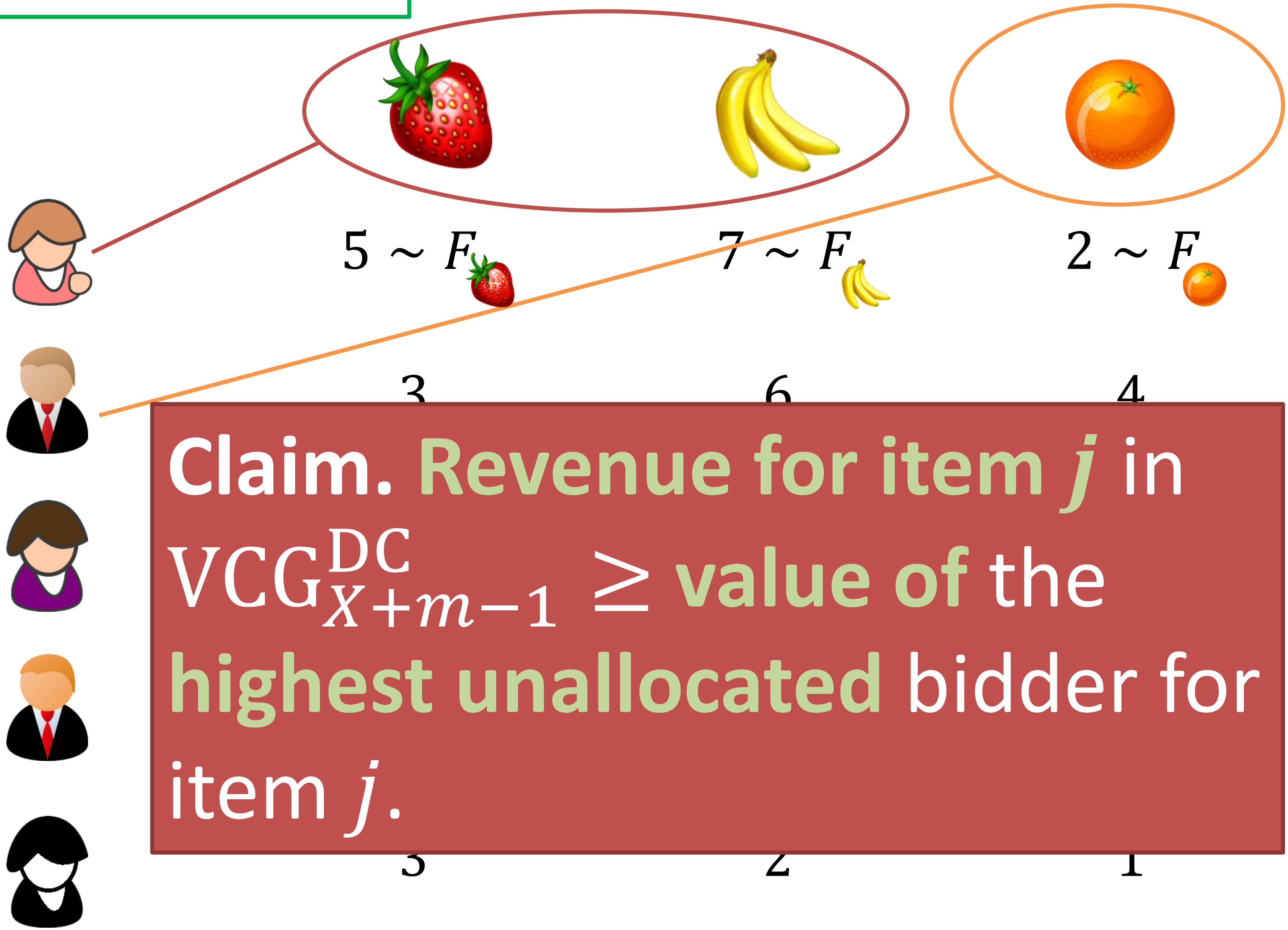
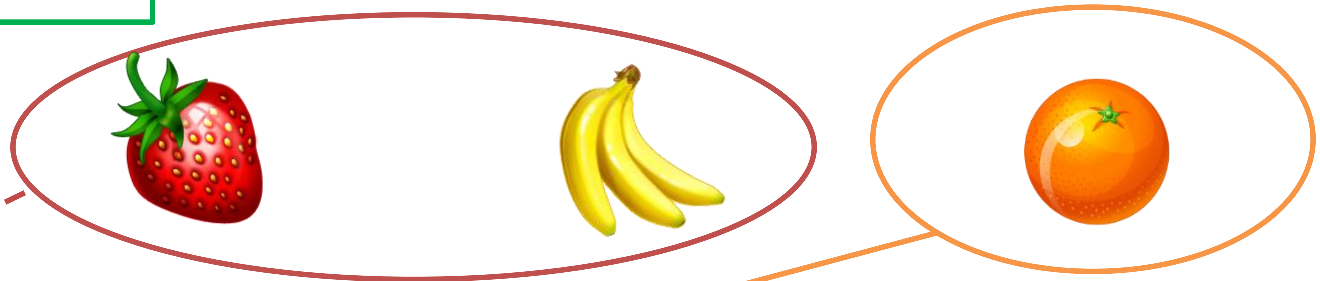
$\mathrm{VCG}_X^{\mathrm{Add}} \leq \mathrm{VCG}_{X+m-1}^{\mathrm{DC}}$



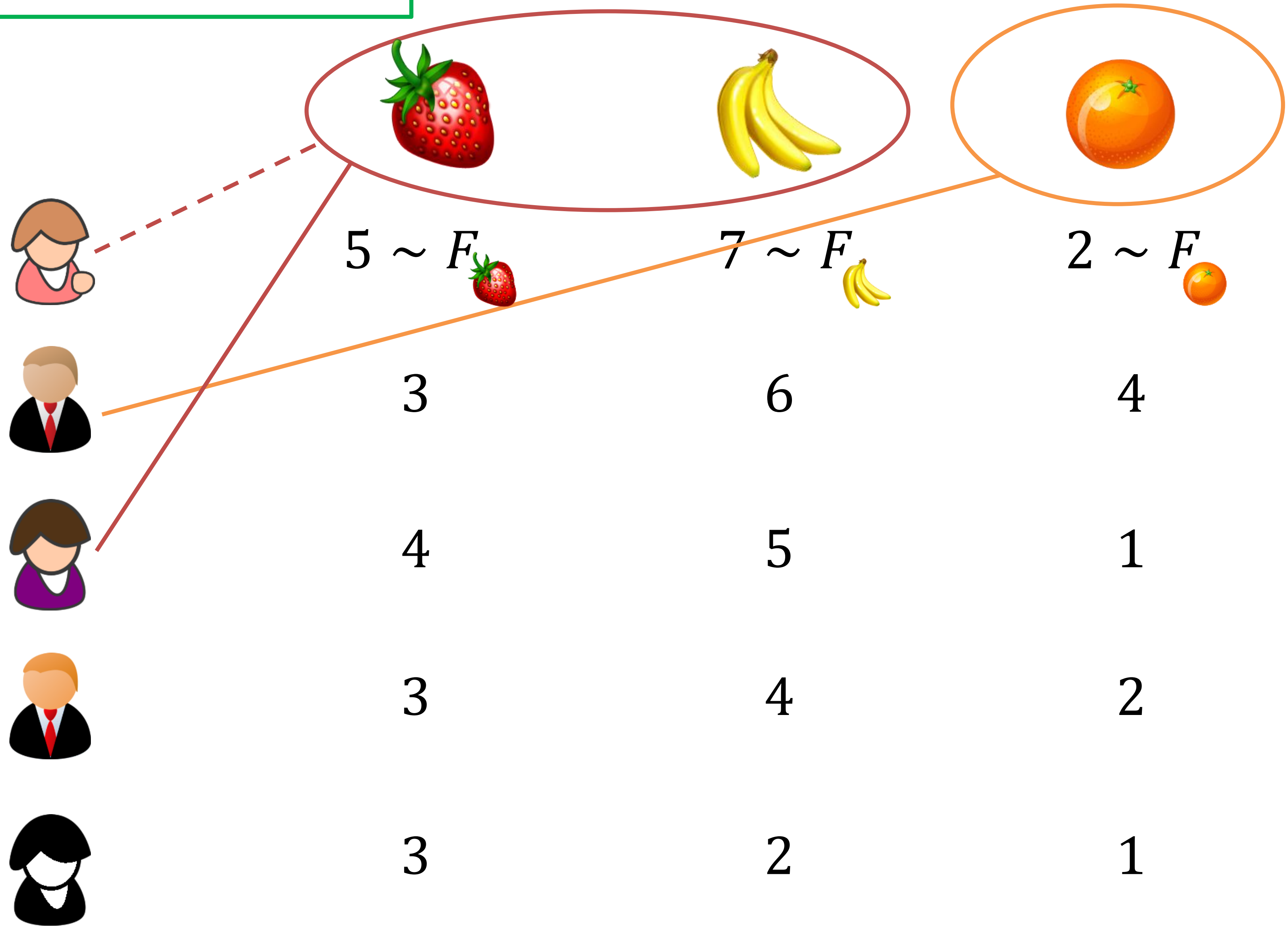
Claim. Revenue for item j in $\mathrm{VCG}_{X+m-1}^{\mathrm{DC}} \geq$ value of the highest unallocated bidder for item j.

$$\mathrm{VCG}_X^{\mathrm{Add}} \leq \mathrm{VCG}_{X+m-1}^{\mathrm{DC}}$$

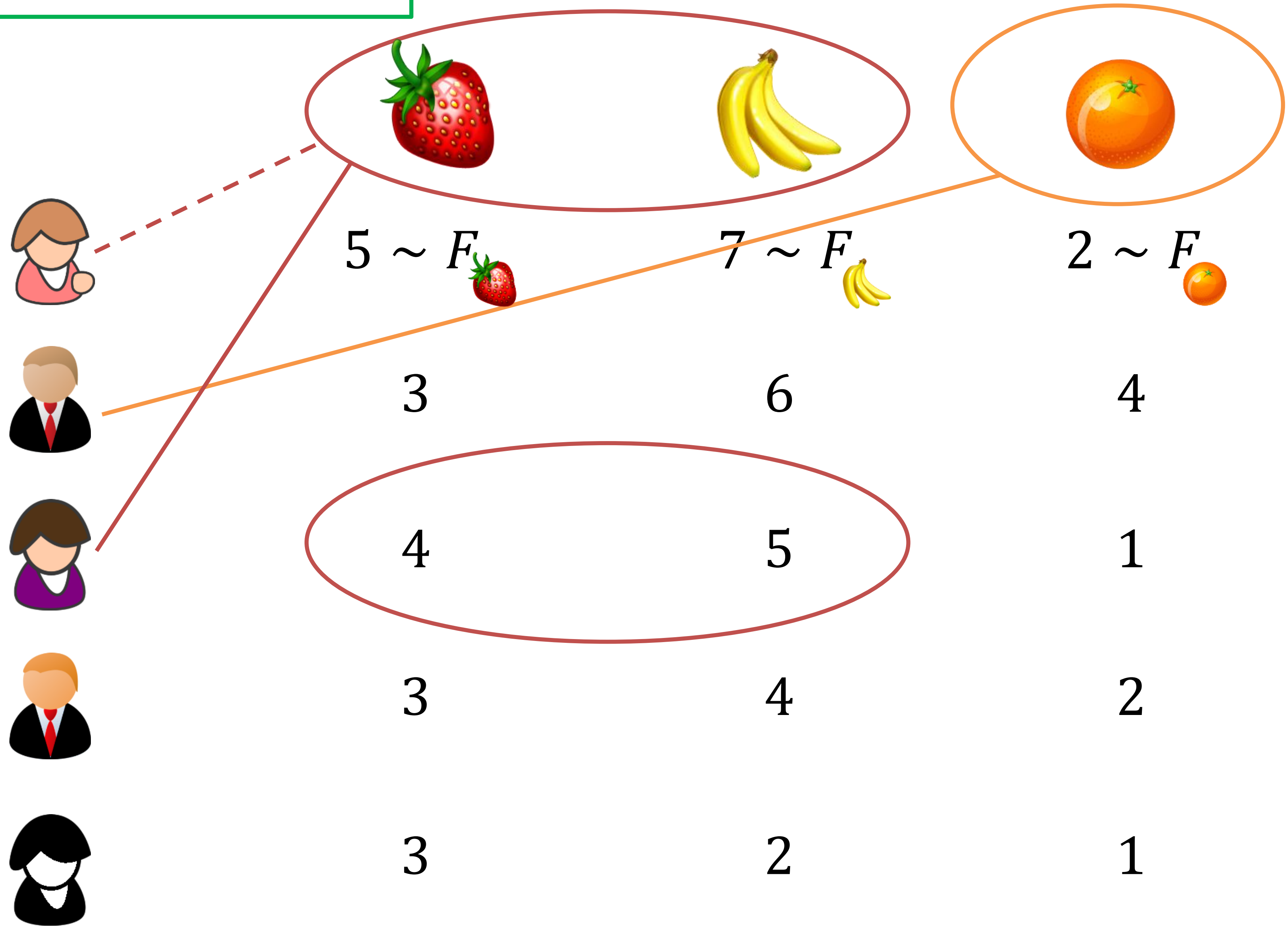


$5 \sim F_{\text{strawberry}}$	$7 \sim F_{\text{banana}}$	$2 \sim F_{\text{orange}}$
3	6	4
4	5	1
3	4	2
3	2	1

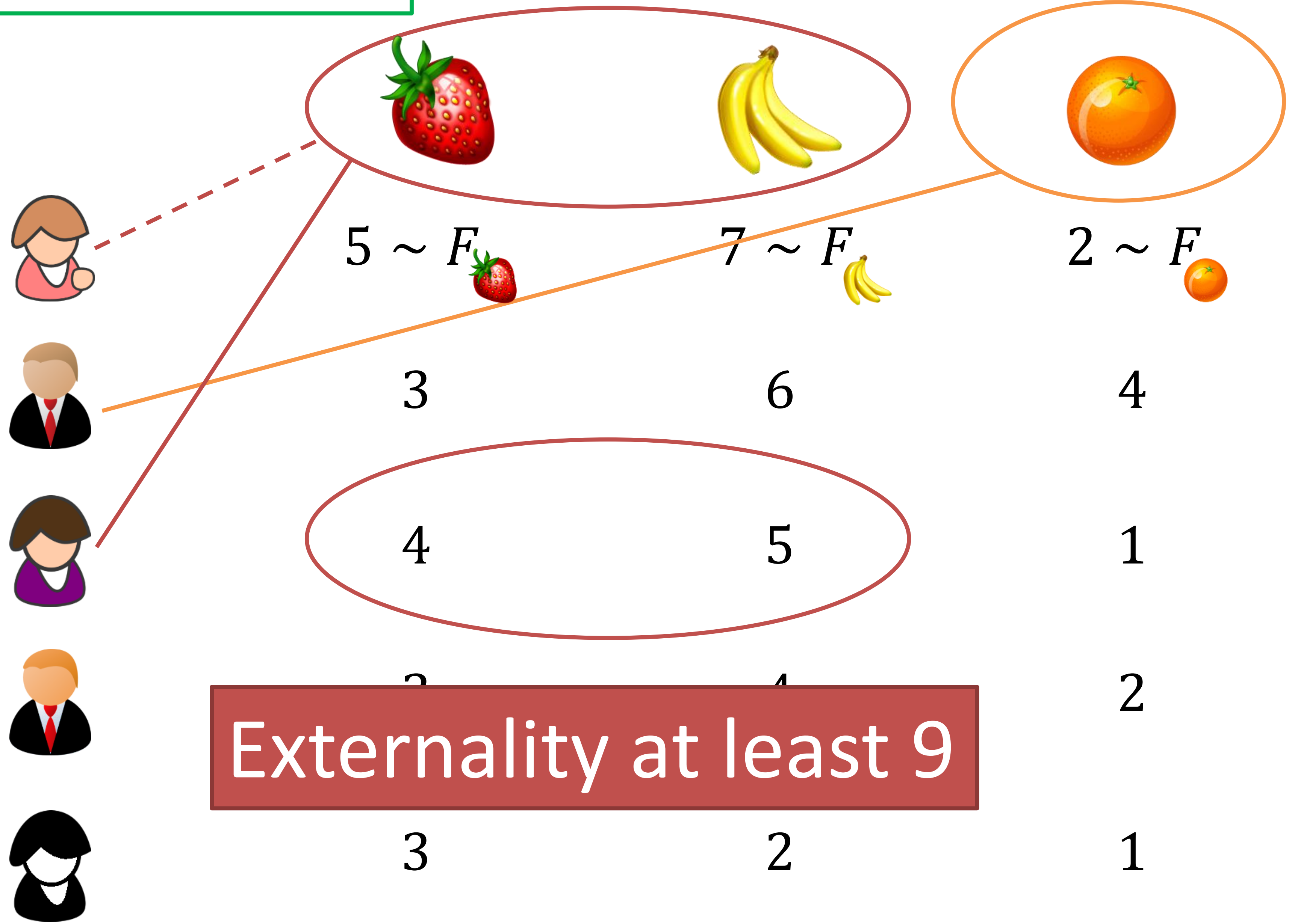
$$\mathrm{VCG}_X^{\mathrm{Add}} \leq \mathrm{VCG}_{X+m-1}^{\mathrm{DC}}$$



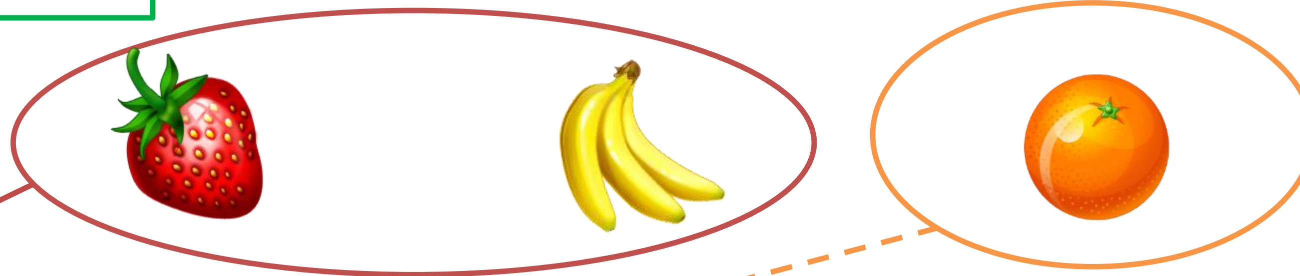
$$\mathrm{VCG}_X^{\mathrm{Add}} \leq \mathrm{VCG}_{X+m-1}^{\mathrm{DC}}$$



$$\mathrm{VCG}_X^{\mathrm{Add}} \leq \mathrm{VCG}_{X+m-1}^{\mathrm{DC}}$$



$$\mathrm{VCG}_{X}^{\mathrm{Add}} \leq \mathrm{VCG}_{X+m-1}^{\mathrm{DC}}$$



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$$\mathrm{VCG}_X^{\mathrm{Add}} \le \mathrm{VCG}_{X+m-1}^{\mathrm{DC}}$$

$5 \sim F$	$7 \sim F$	$2 \sim F$
3	6	4
4	5	1
3	4	2
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$\mathrm{VCG}_{X}^{\mathrm{Add}} \leq \mathrm{VCG}_{X+m-1}^{\mathrm{DC}}$

$5 \sim F$	$7 \sim F$	$2 \sim F$
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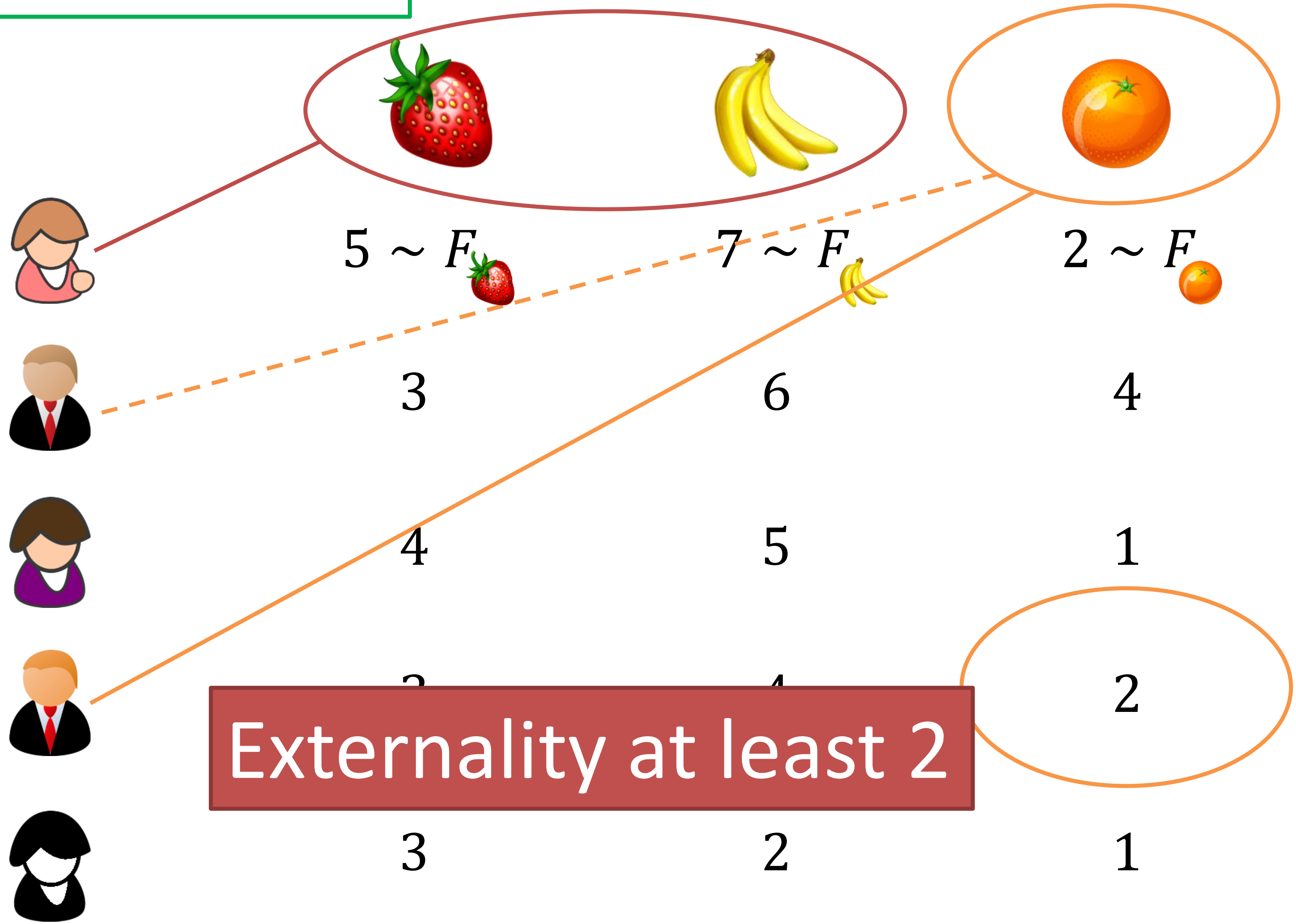
$$\mathrm{VCG}_X^{\mathrm{Add}} \leq \mathrm{VCG}_{X+m-1}^{\mathrm{DC}}$$



$$\mathrm{VCG}_X^{\mathrm{Add}} \leq \mathrm{VCG}_{X+m-1}^{\mathrm{DC}}$$

$$\mathrm{VCG}_X^{\mathrm{Add}} \leq \mathrm{VCG}_{X+m-1}^{\mathrm{DC}}$$

$\mathrm{VCG}_X^{\mathrm{Add}}(j) =$ 2nd highest of X samples from F_j $\leq$ Highest value of unallocated bidder for j $\leq \mathrm{VCG}_{X+m-1}^{\mathrm{DC}}(j)$

$$\mathrm{VCG}_X^{\mathrm{Add}} \leq \mathrm{VCG}_{X+m-1}^{\mathrm{DC}}$$

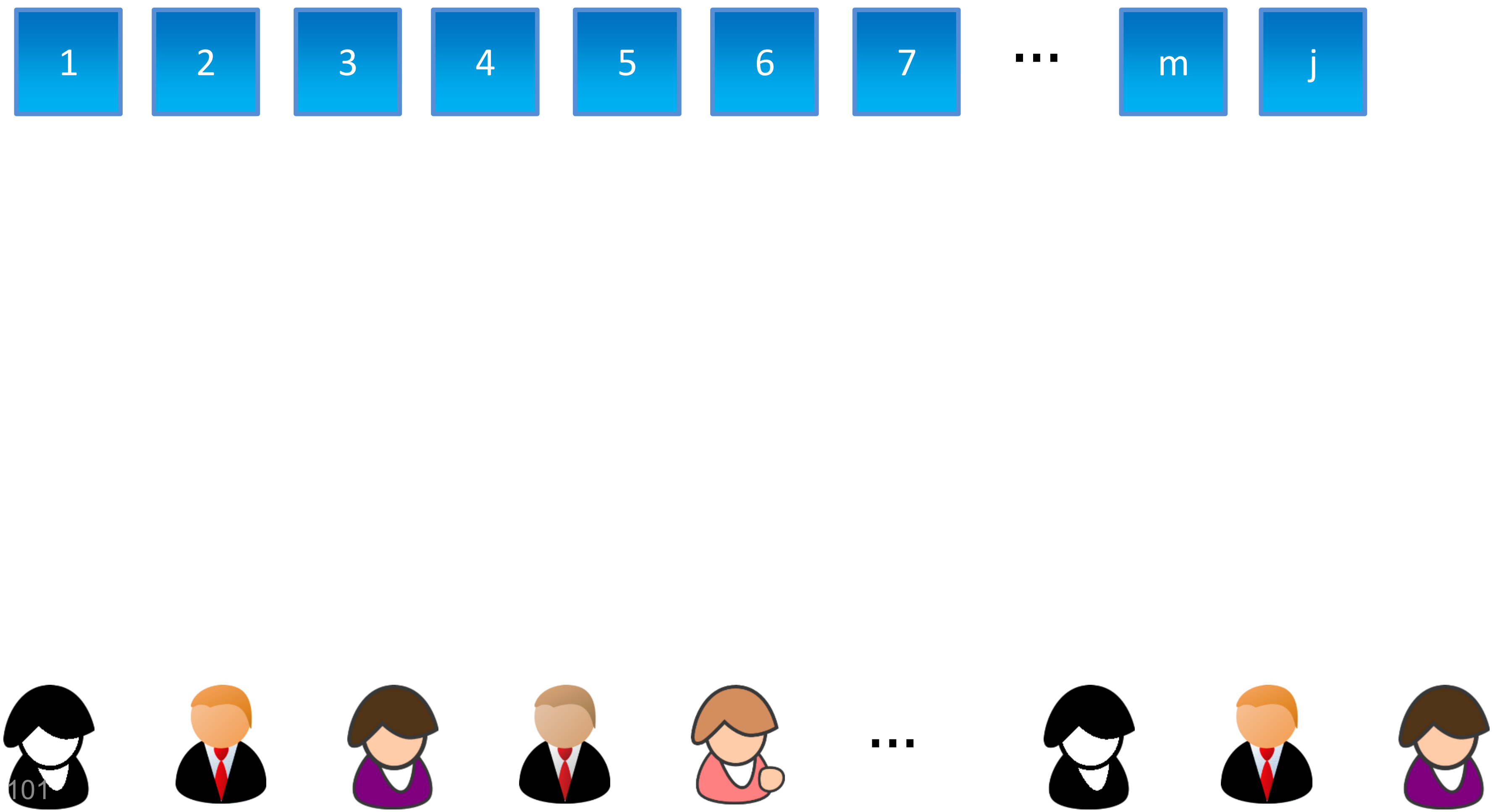
$\mathrm{VCG}_X^{\mathrm{Add}}(j) =$ 2nd highest of X samples from F_j $\leq$ Highest value of unallocated bidder for j $\leq \mathrm{VCG}_{X+m-1}^{\mathrm{DC}}(j)$

$$\mathrm{VCG}_X^{\mathrm{Add}} \leq \mathrm{VCG}_{X+m-1}^{\mathrm{DC}}$$

$\mathrm{VCG}_X^{\mathrm{Add}}(j) =$ 2[nd] highest of X samples from F_j $\leq$ Highest value of unallocated bidder for j $\leq \mathrm{VCG}_{X+m-1}^{\mathrm{DC}}(j)$

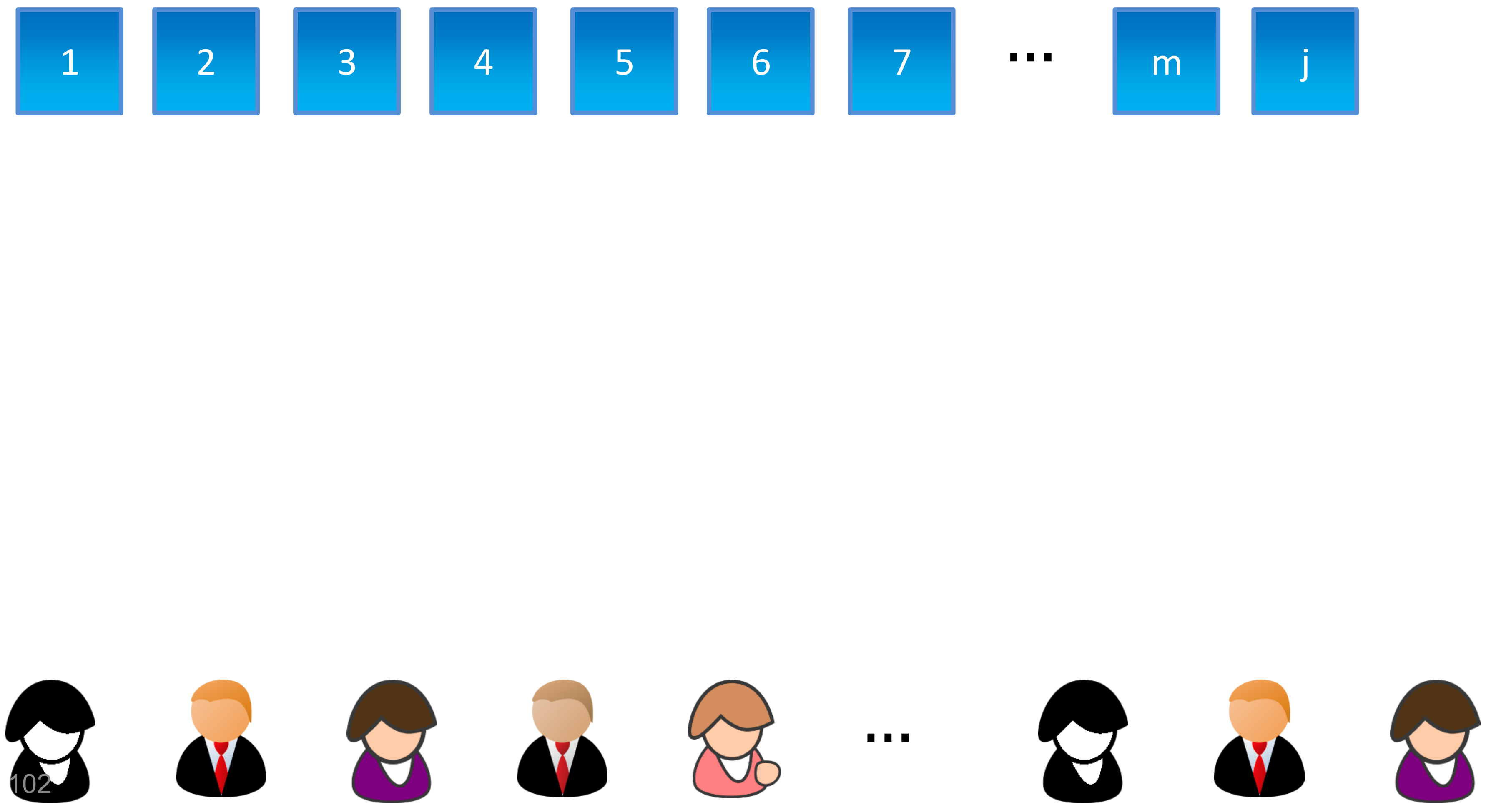
Identify X bidders in $\mathrm{VCG}_{X+m-1}^{\mathrm{DC}}$ before sampling their value for item j out of which at most one will be allocated anything

$$\mathrm{VCG}_X^{\mathrm{Add}} \leq \mathrm{VCG}_{X+m-1}^{\mathrm{DC}}$$



$\text{VCG}_X^{\text{Add}} \leq \text{VCG}_{X+m-1}^{\text{DC}}$

(Assume wlog unique optimal allocation)



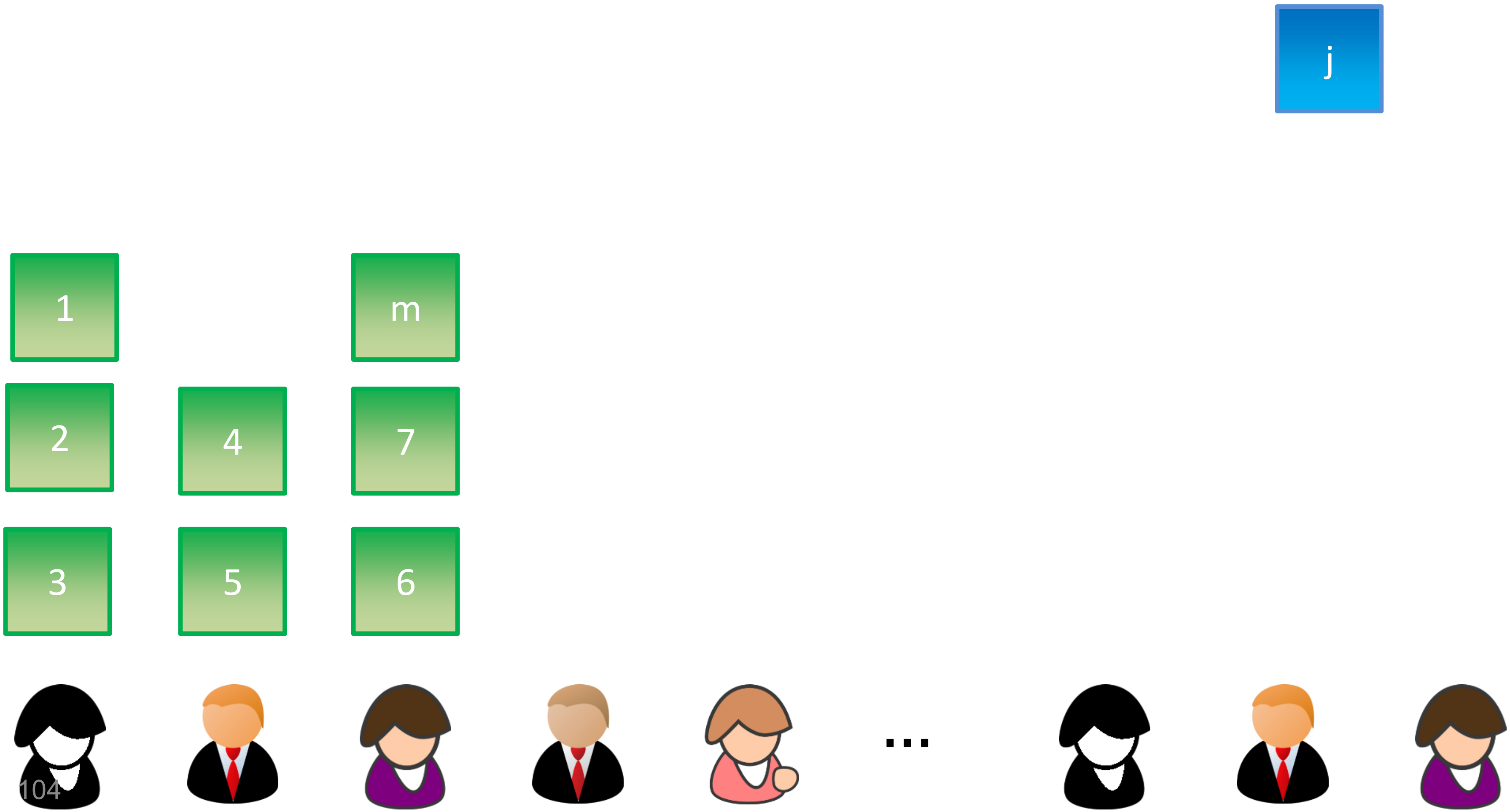
$$\mathrm{VCG}_{X}^{\mathrm{Add}} \leq \mathrm{VCG}_{X+m-1}^{\mathrm{DC}}$$

(Assume wlog unique optimal allocation)

1. Sample valuations for all items but *j*.

$\mathrm{VCG}_X^{\mathrm{Add}} \leq \mathrm{VCG}_{X+m-1}^{\mathrm{DC}}$ (Assume wlog unique optimal allocation)

2. Compute an optimal allocation without item *j*.

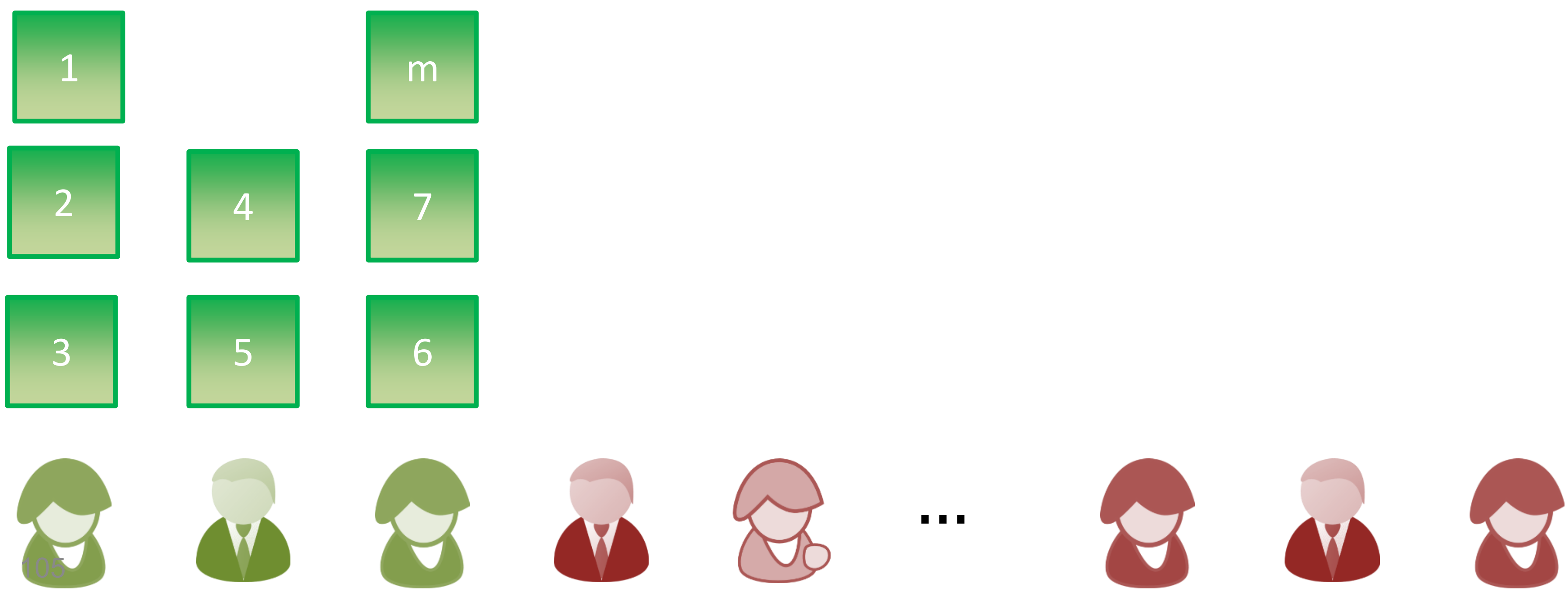


$\text{VCG}_X^{\text{Add}} \leq \text{VCG}_{X+m-1}^{\text{DC}}$ (Assume wlog unique optimal allocation)

2. Compute an optimal allocation without item j.

Set A of allocated bidders

Set $\bar{A}$ of unallocated bidders



$\text{VCG}_X^{\text{Add}} \leq \text{VCG}_{X+m-1}^{\text{DC}}$

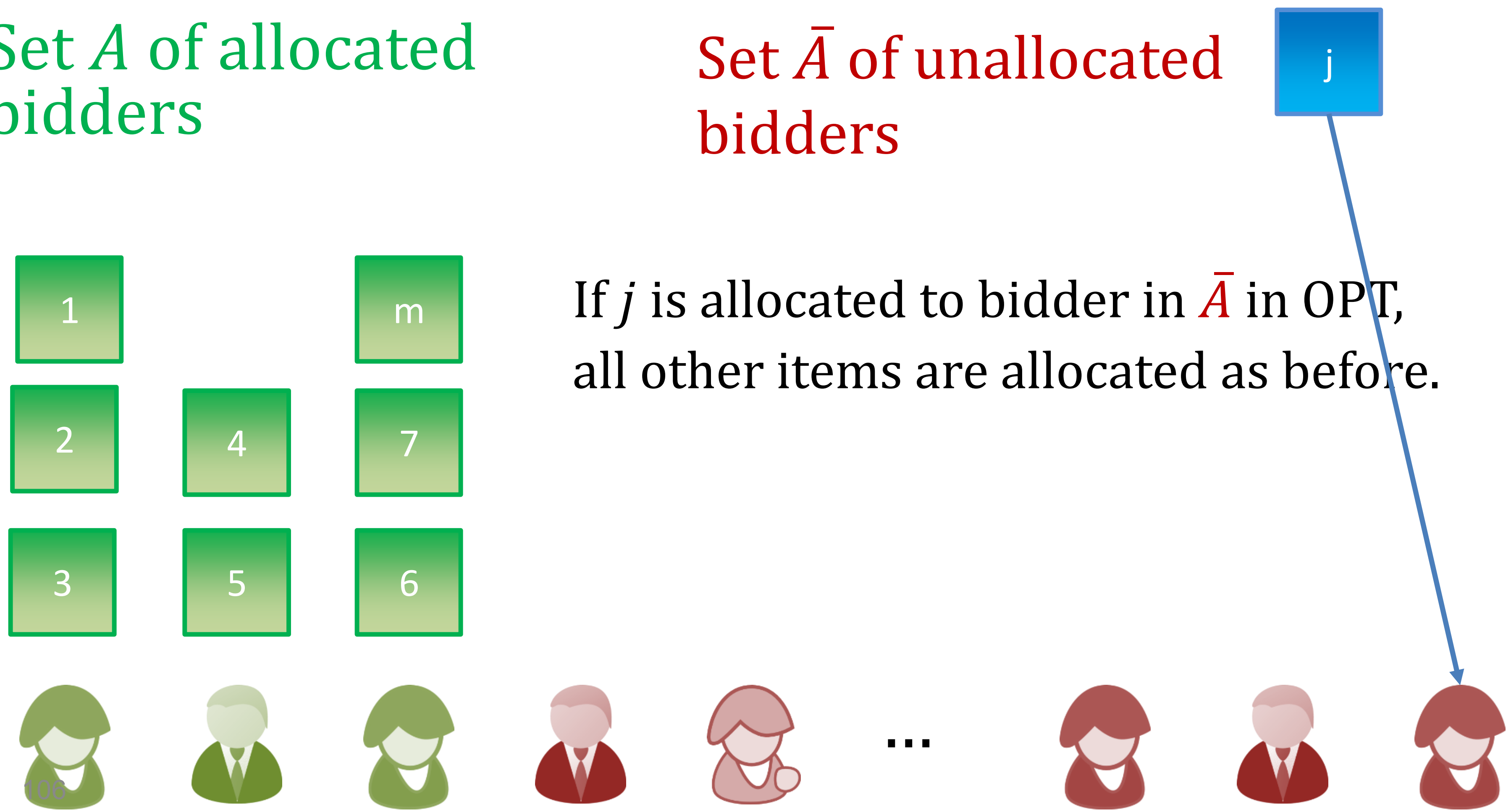
(Assume wlog unique optimal allocation)

2. Compute an optimal allocation without item *j*.

Set A of allocated bidders

Set $\bar{A}$ of unallocated bidders

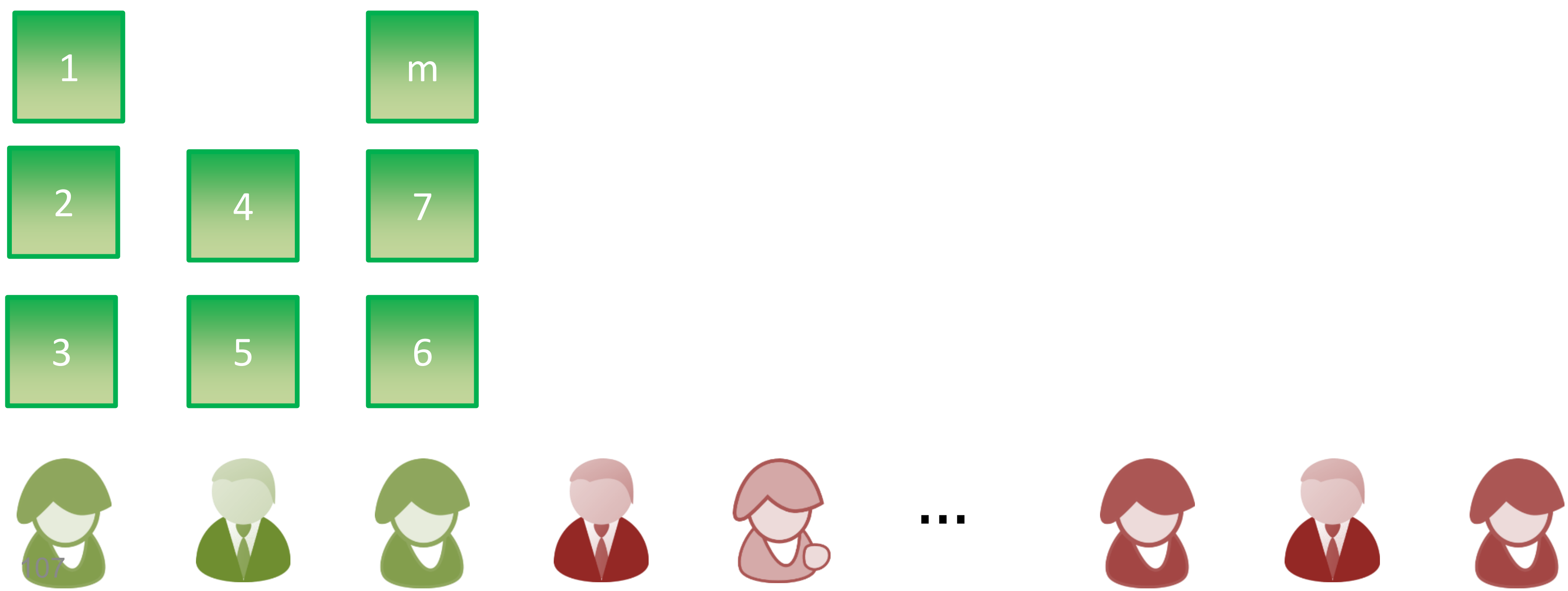
If j is allocated to bidder in $\bar{A}$ in OPT, all other items are allocated as before.



$$\boxed{\mathrm{VCG}_X^{\mathrm{Add}} \leq \mathrm{VCG}_{X+m-1}^{\mathrm{DC}}}$$

(Assume wlog unique optimal allocation)

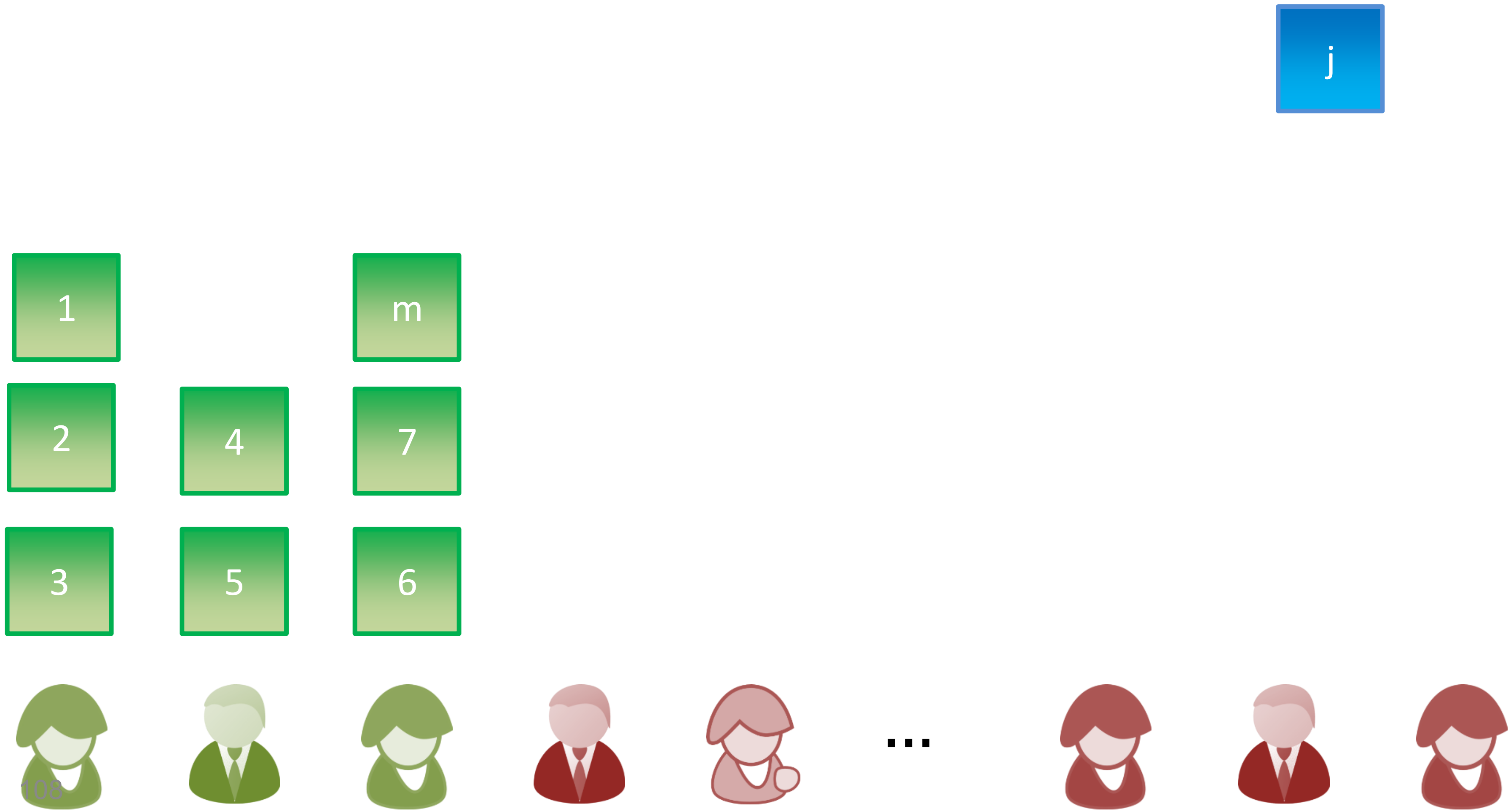
3. Sample values for j for agents in A and compute the optimal allocation where j is allocated to a bidder in A .



$$\text{VCG}_X^{\text{Add}} \leq \text{VCG}_{X+m-1}^{\text{DC}}$$

(Assume wlog unique optimal allocation)

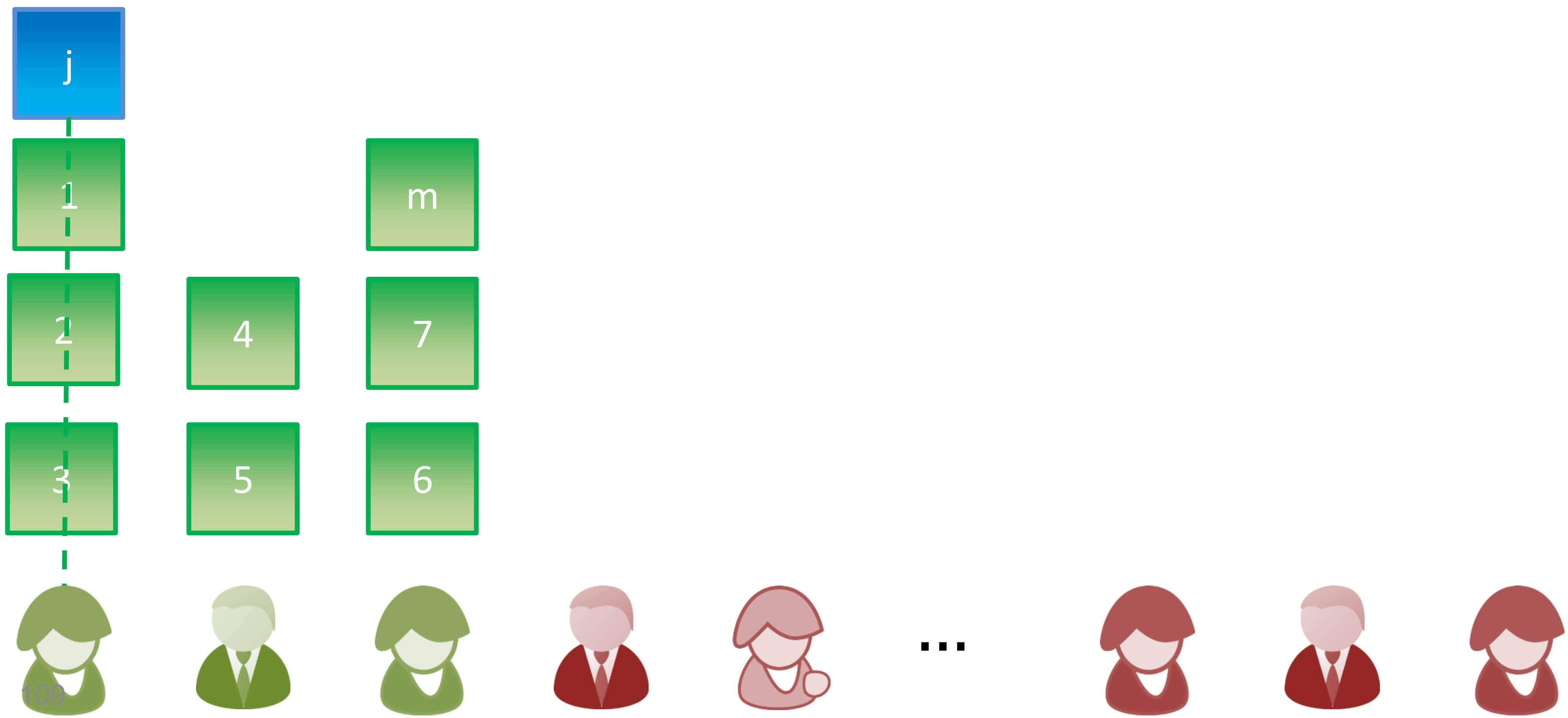
3. Compute $\text{OPT}_{j \in A}$



$\mathrm{VCG}_X^{\mathrm{Add}} \leq \mathrm{VCG}_{X+m-1}^{\mathrm{DC}}$ (Assume wlog unique optimal allocation)

3. Compute $\mathrm{OPT}_{j\in A}$

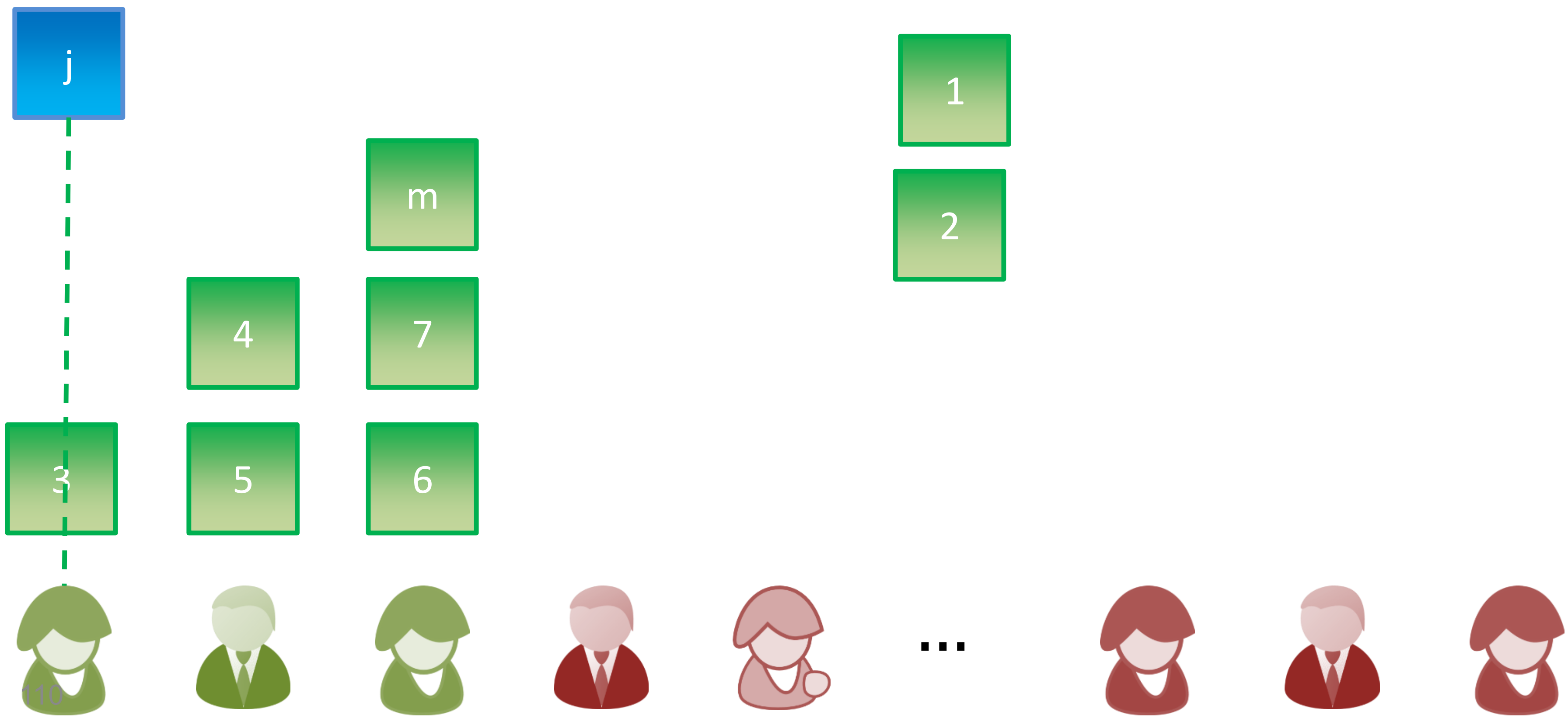
Some items might be vacated due to feasibility



$\mathrm{VCG}_X^{\mathrm{Add}} \leq \mathrm{VCG}_{X+m-1}^{\mathrm{DC}}$ (Assume wlog unique optimal allocation)

3. Compute $\mathrm{OPT}_{j \in A}$

Some items might be snatched from other agents

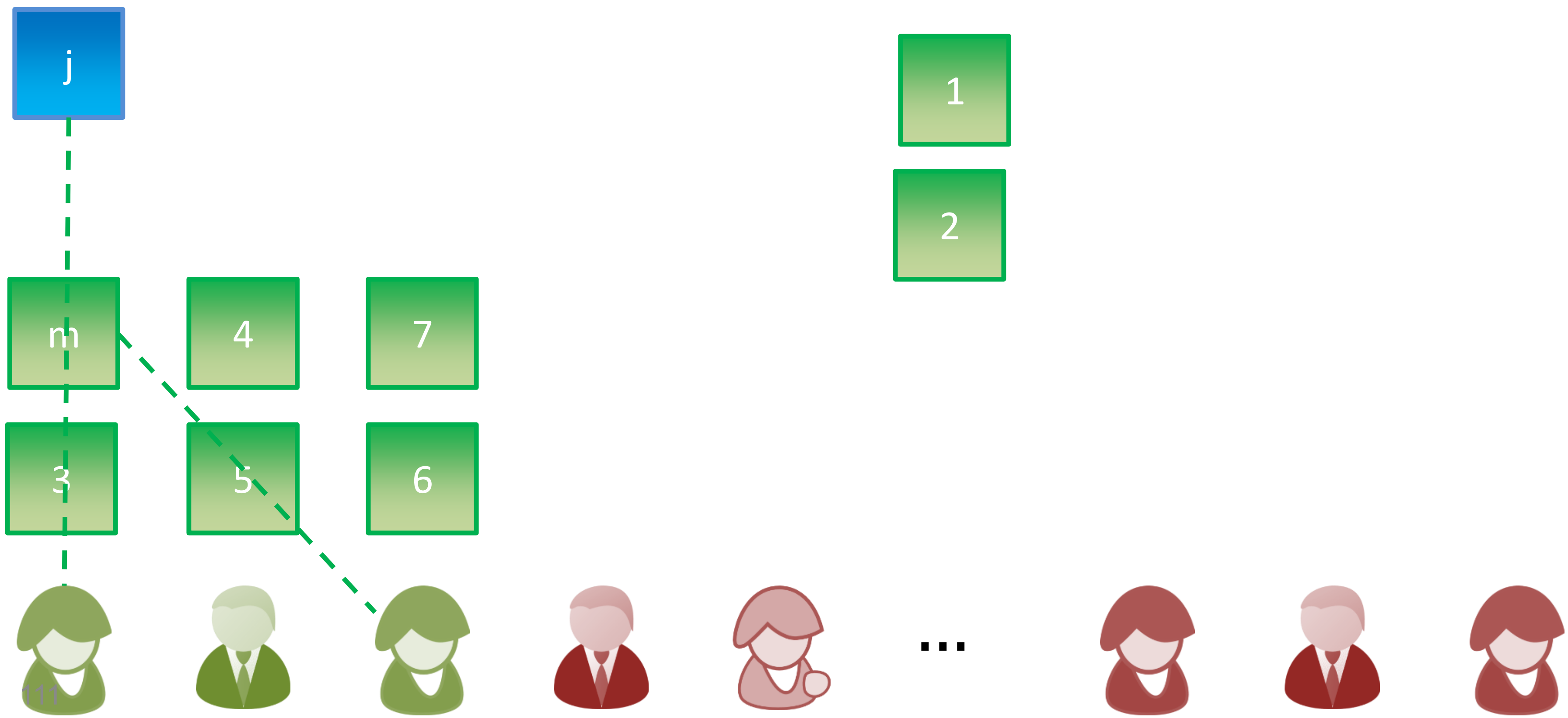


$$\mathrm{VCG}_X^{\mathrm{Add}} \leq \mathrm{VCG}_{X+m-1}^{\mathrm{DC}}$$

(Assume wlog unique optimal allocation)

3. Compute $\mathrm{OPT}_{j \in A}$

Continue with this process

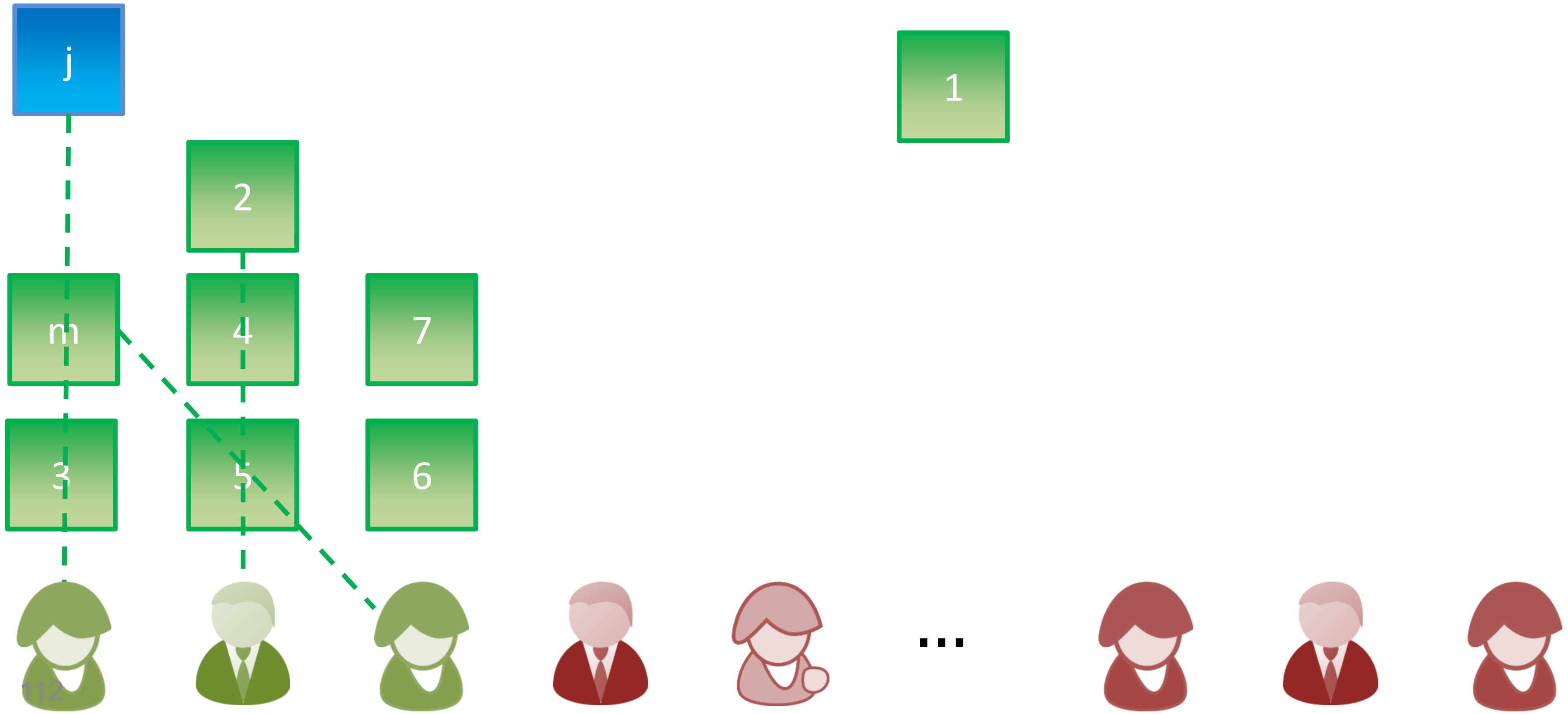


$$\boxed{\mathrm{VCG}_X^{\mathrm{Add}} \leq \mathrm{VCG}_{X+m-1}^{\mathrm{DC}}}$$

(Assume wlog unique optimal allocation)

3. Compute $\mathrm{OPT}_{j\in A}$

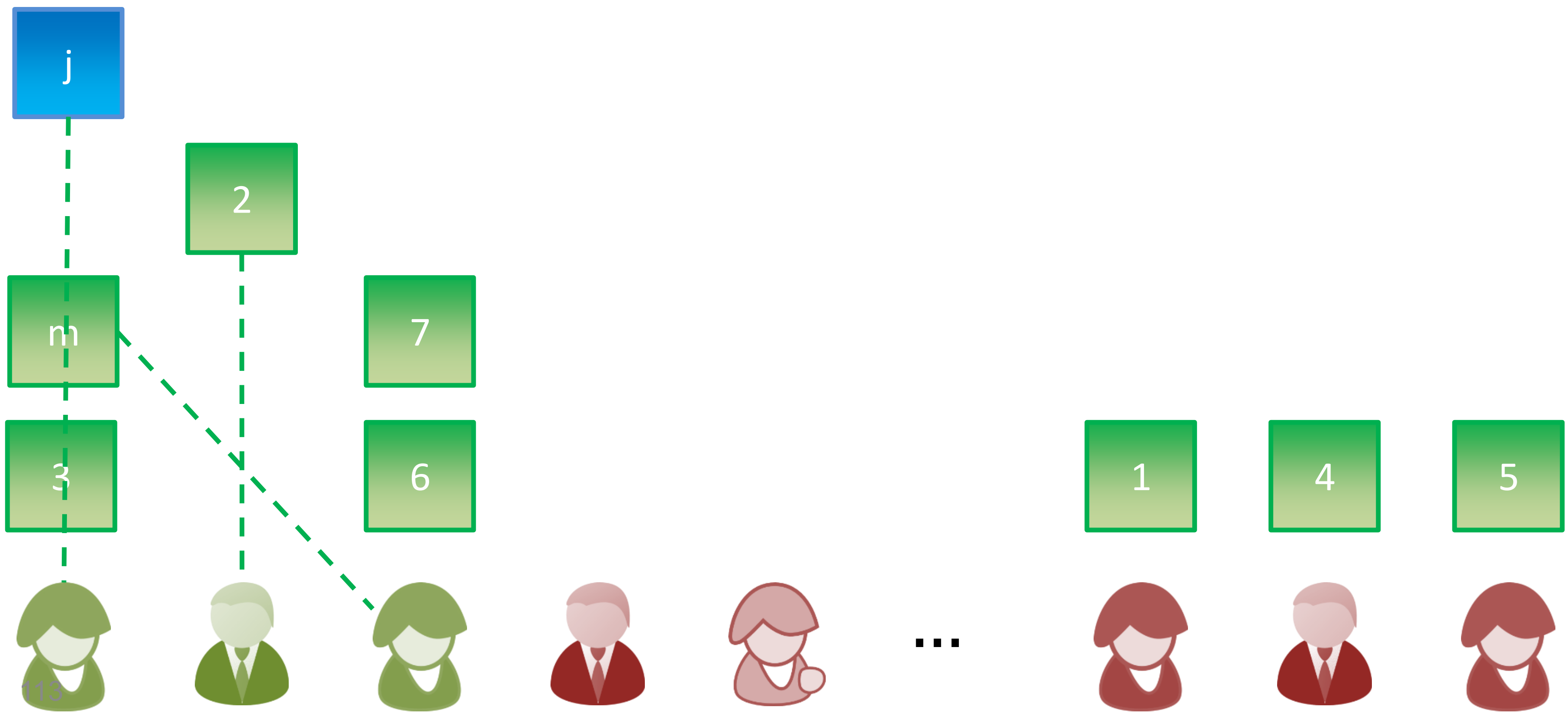
Continue with this process



$$\mathrm{VCG}_X^{\mathrm{Add}} \leq \mathrm{VCG}_{X+m-1}^{\mathrm{DC}}$$

(Assume wlog unique optimal allocation)

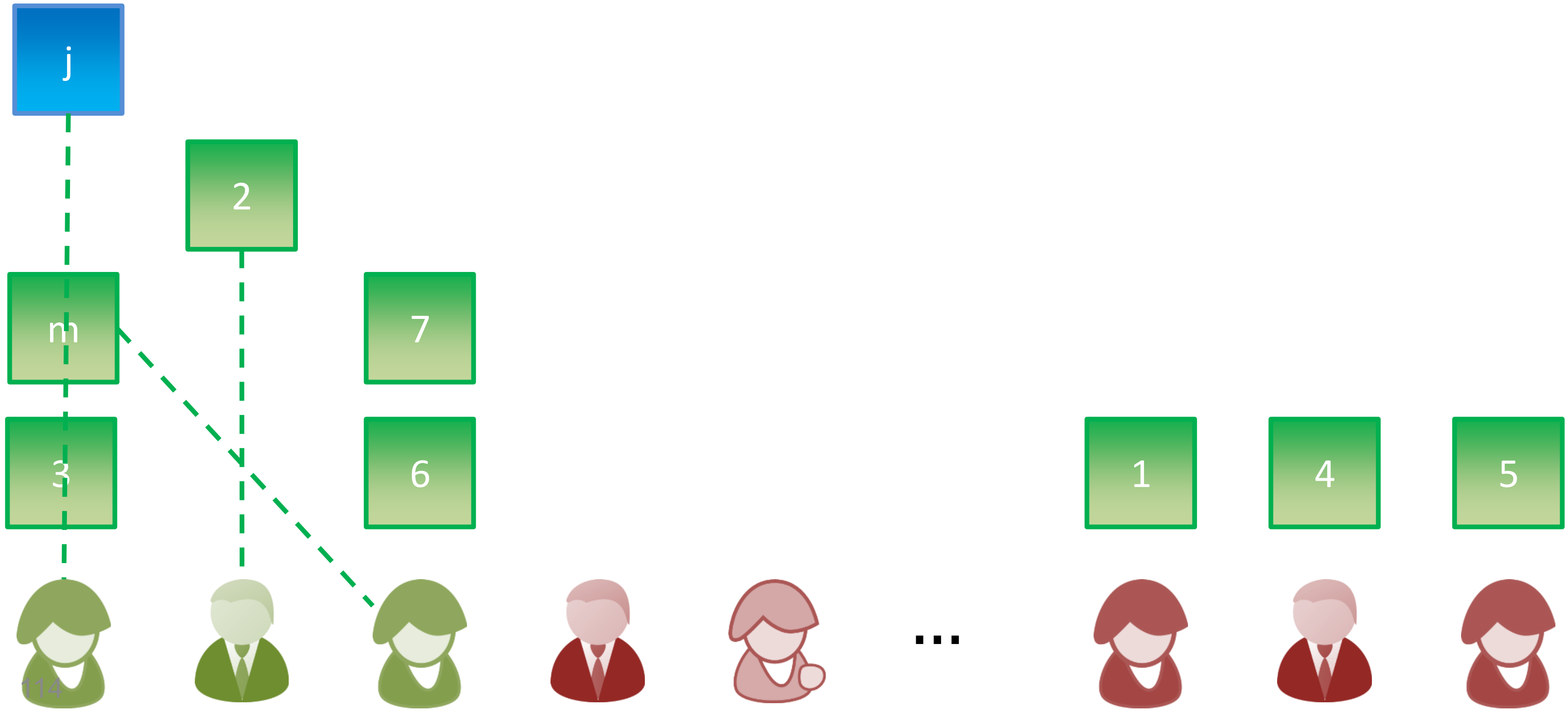
3. Compute $\mathrm{OPT}_{j \in A}$. There are $\geq |A|$ items allocated to agents in A.



$\text{VCG}_X^{\text{Add}} \leq \text{VCG}_{X+m-1}^{\text{DC}}$ (Assume wlog unique optimal allocation)

3. Compute $\text{OPT}_{j \in A}$. There are $\geq |A|$ items allocated to agents in A.

– Map each agent who's item was snatched to the snatched item.

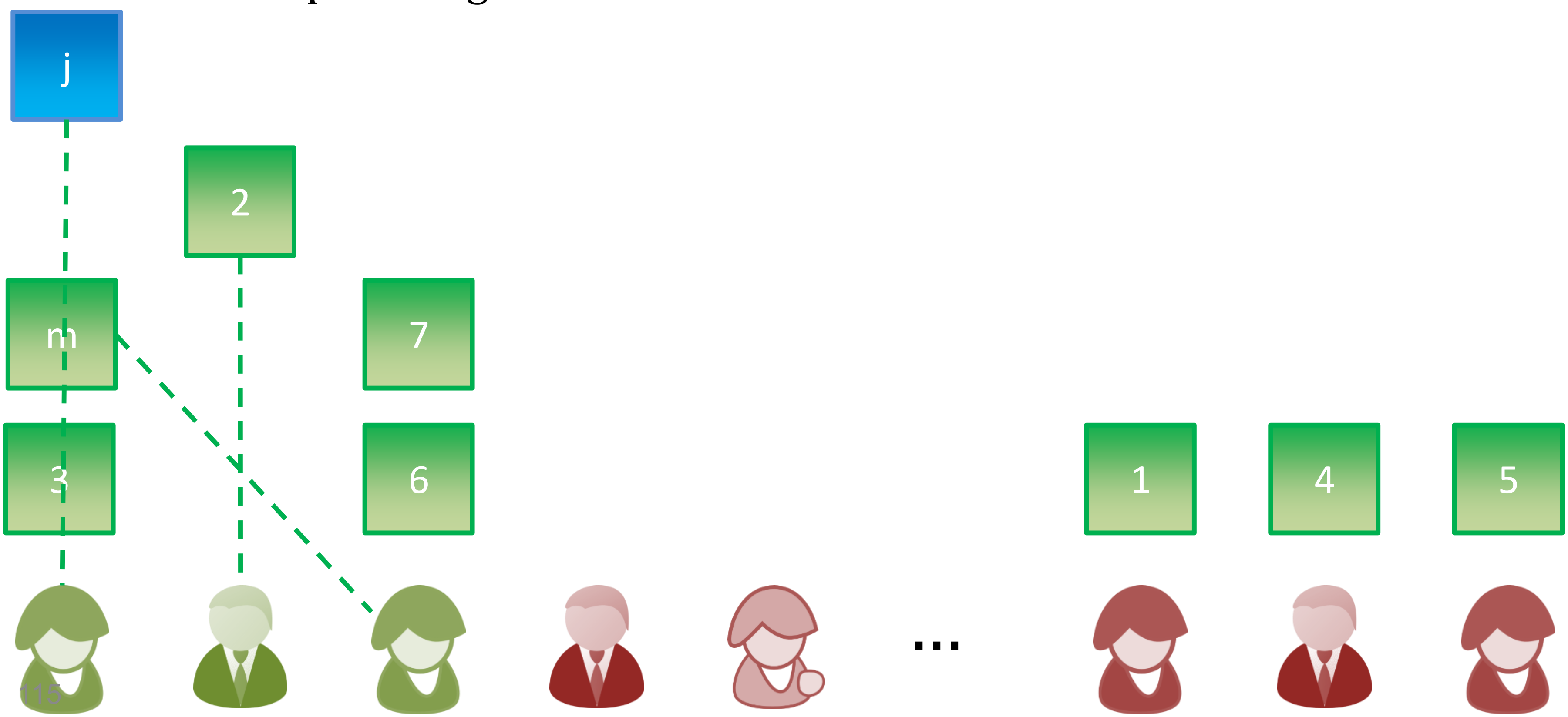


$$\mathrm{VCG}_X^{\mathrm{Add}} \leq \mathrm{VCG}_{X+m-1}^{\mathrm{DC}}$$

(Assume wlog unique optimal allocation)

3. Compute $\mathrm{OPT}_{j \in A}$. There are $\geq |A|$ items allocated to agents in A.

- Map each agent who's item was snatched to the snatched item.
- Map each agent who took a vacated item to the item.

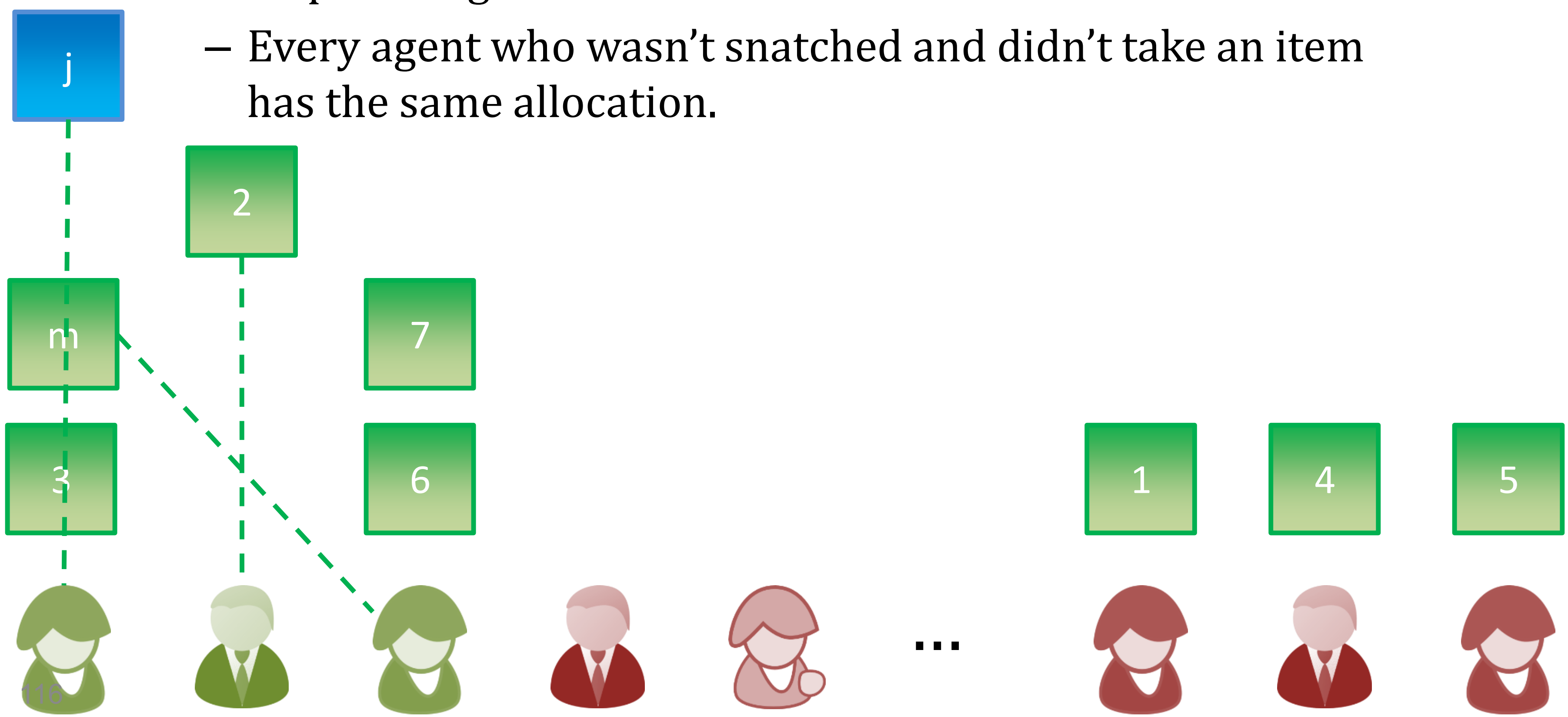


$\mathrm{VCG}_X^{\mathrm{Add}} \leq \mathrm{VCG}_{X+m-1}^{\mathrm{DC}}$

(Assume wlog unique optimal allocation)

3. Compute $\mathrm{OPT}_{j \in A}$. There are $\geq |A|$ items allocated to agents in A.

- Map each agent who's item was snatched to the snatched item.
- Map each agent who took a vacated item to the item.
- Every agent who wasn't snatched and didn't take an item has the same allocation.



$\text{VCG}_X^{\text{Add}} \leq \text{VCG}_{X+m-1}^{\text{DC}}$ (Assume wlog unique optimal allocation)

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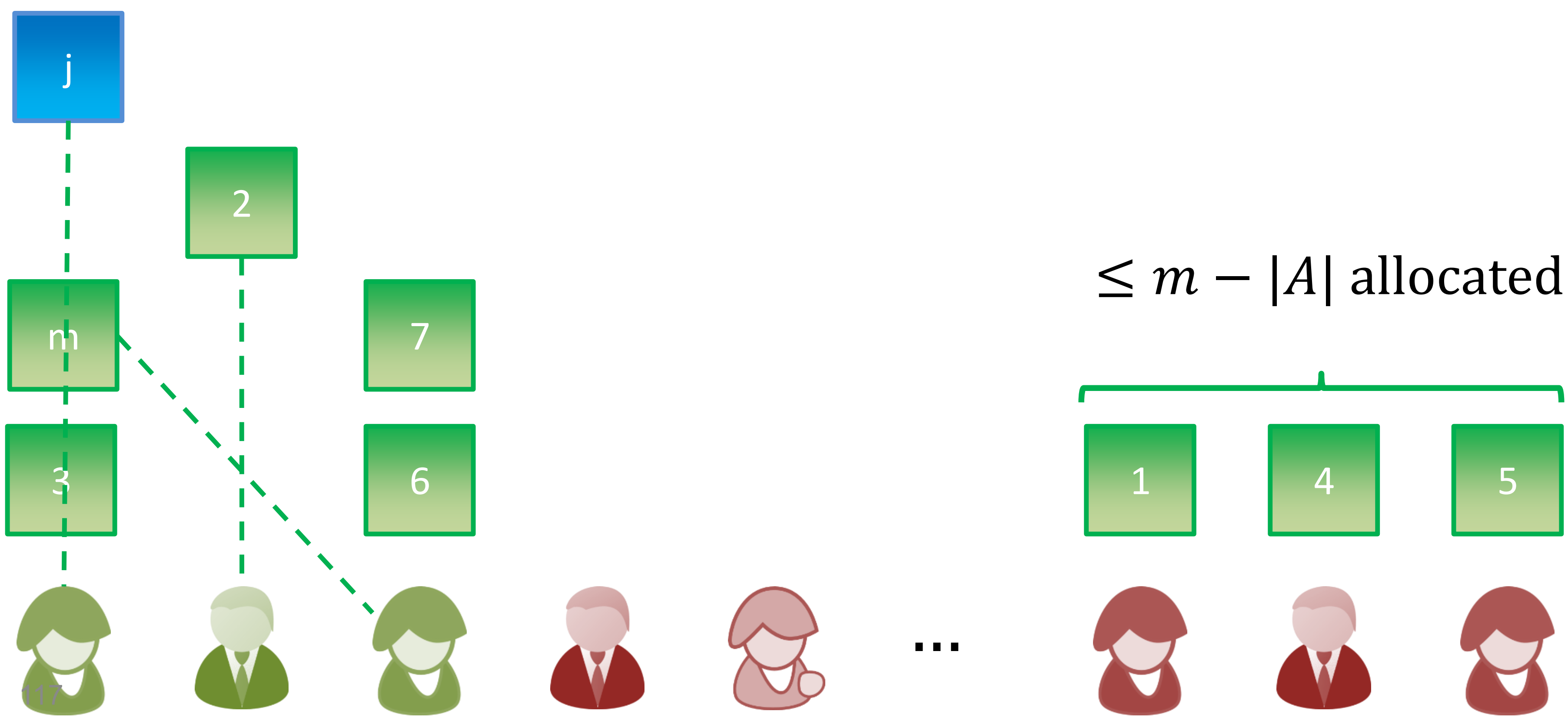

$\mathrm{VCG}_X^{\mathrm{Add}} \leq \mathrm{VCG}_{X+m-1}^{\mathrm{DC}}$

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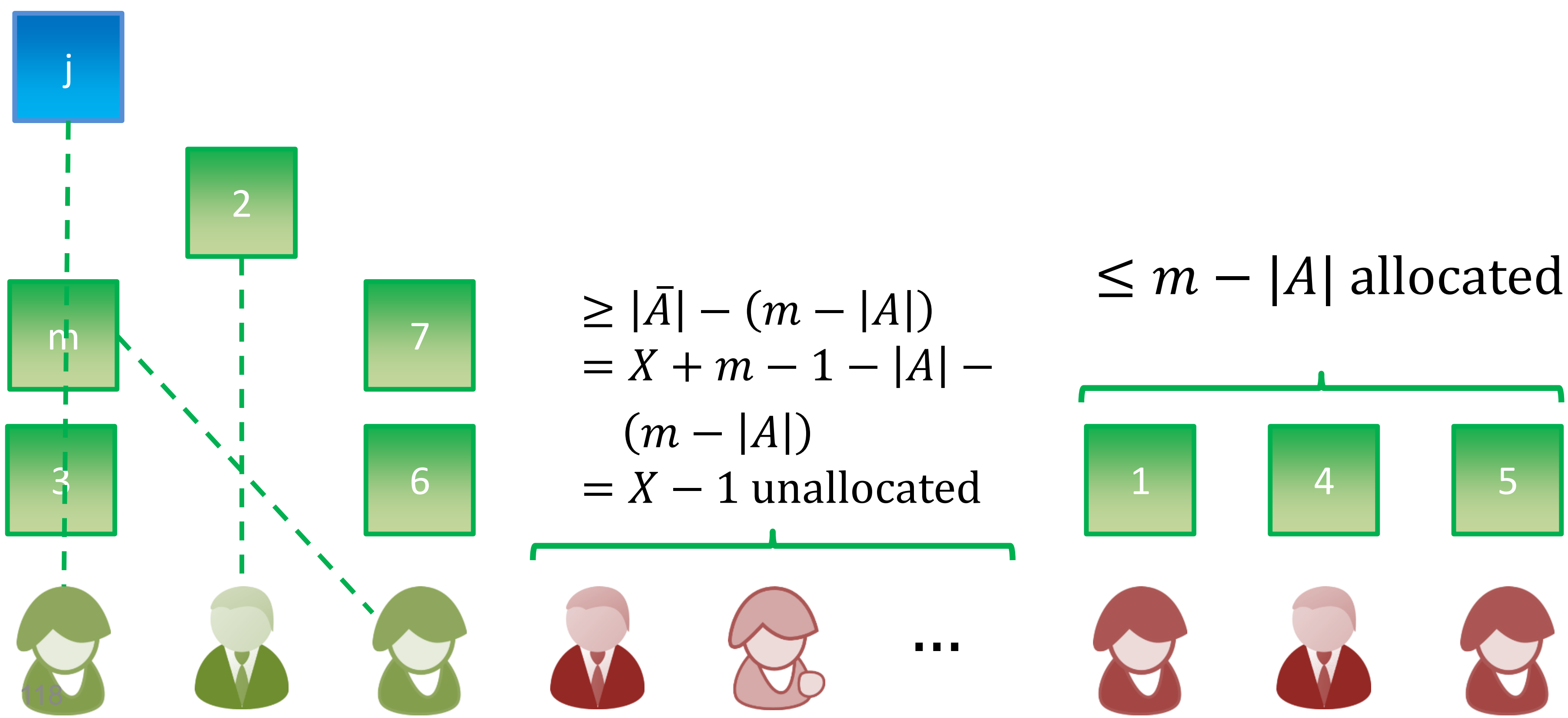
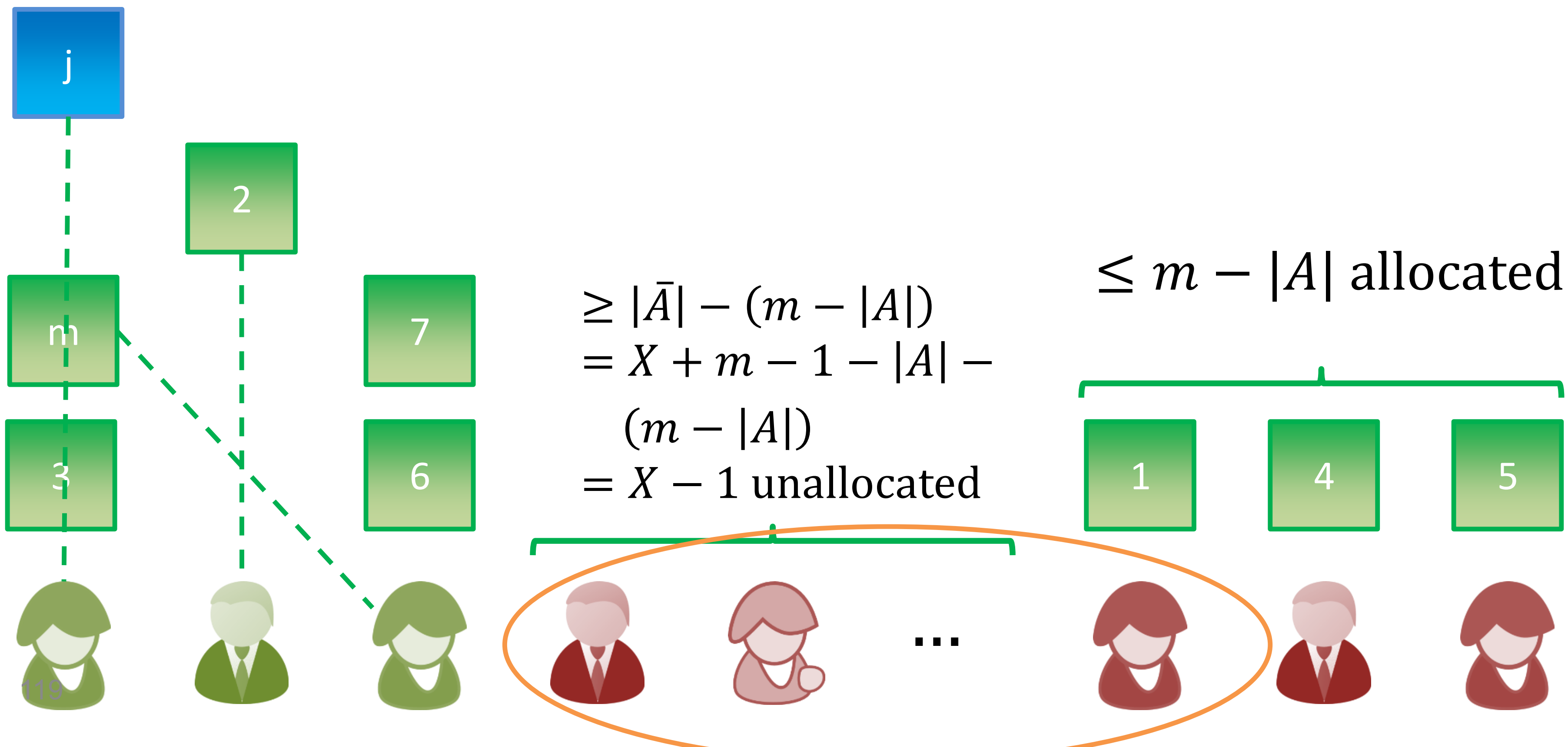
3. Compute $\mathrm{OPT}_{j \in A}$. There are $\geq |A|$ items allocated to agents in A.



$$\mathrm{VCG}_X^{\mathrm{Add}} \le \mathrm{VCG}_{X+m-1}^{\mathrm{DC}}$$

(Assume wlog unique optimal allocation)



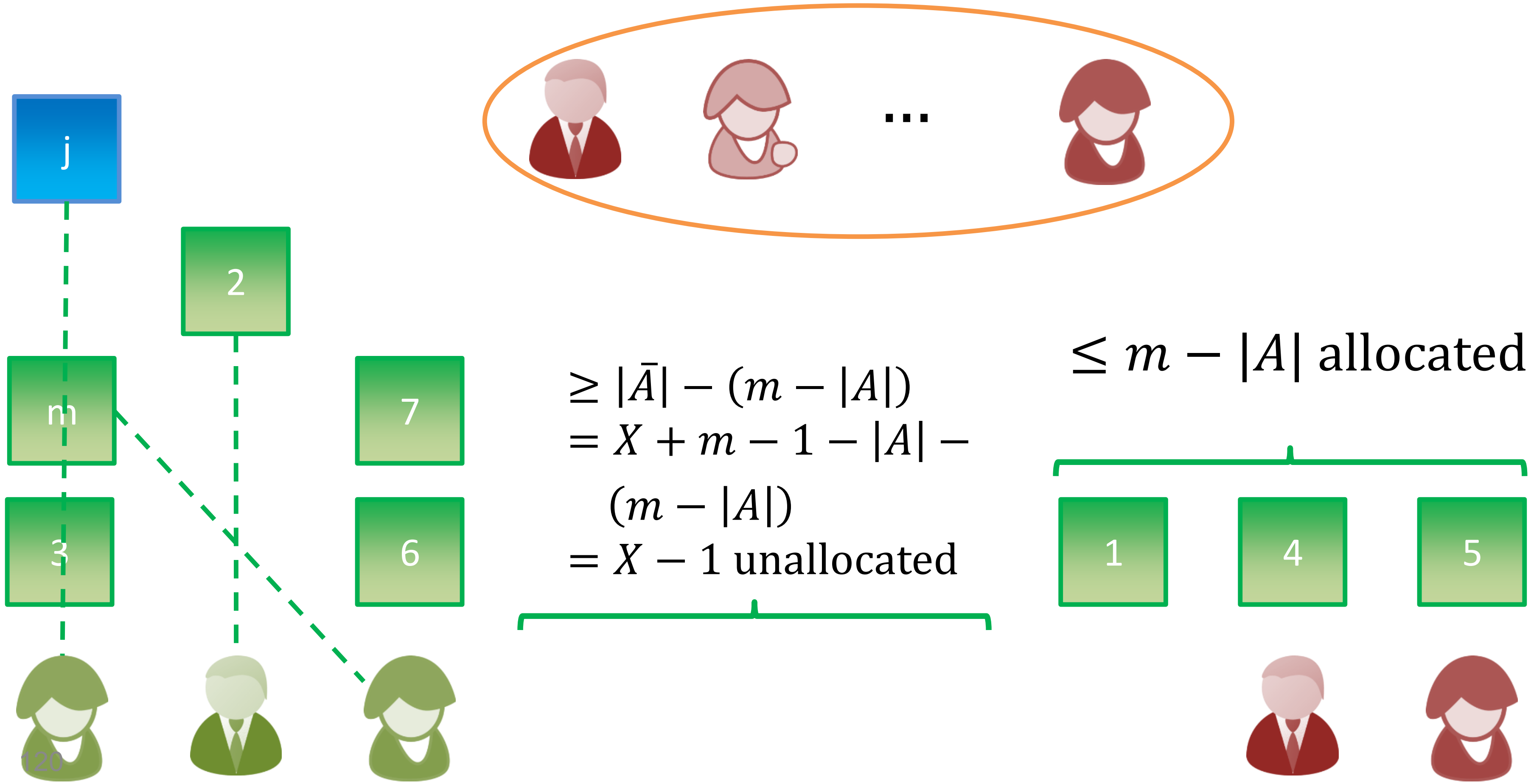
$\ge |\bar{A}| - (m - |A|)$
$= X + m - 1 - |A| - (m - |A|)$
$= X - 1$ unallocated

$\le m - |A|$ allocated

$\mathrm{VCG}_X^{\mathrm{Add}} \leq \mathrm{VCG}_{X+m-1}^{\mathrm{DC}}$ (Assume wlog unique optimal allocation)

X bidders whose values for j are i.i.d. samples from F_j.

At most one is allocated by $\mathrm{VCG}_{X+m-1}^{\mathrm{DC}}$.



$\mathrm{VCG}_X^{\mathrm{Add}} \leq \mathrm{VCG}_{X+m-1}^{\mathrm{DC}}$ (Assume wlog unique optimal allocation)

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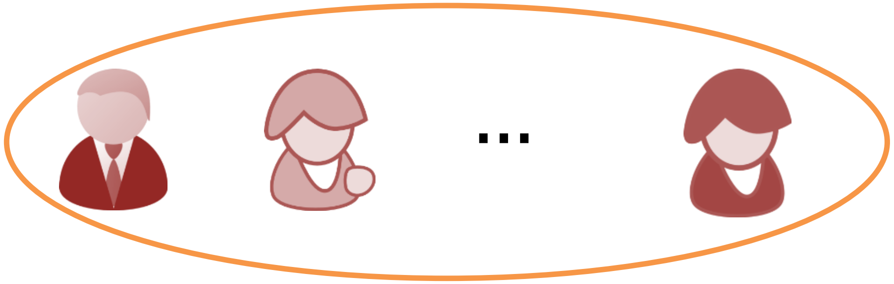

$\mathrm{VCG}_X^{\mathrm{Add}}(j) =$ 2nd highest of X samples from F_j $\leq$ Highest value of unallocated bidder for j $\leq \mathrm{VCG}_{X+m-1}^{\mathrm{DC}}(j)$

Extension to downward closed constraints

$$\mathrm{Rev}_n^{\mathrm{DC}} \leq \mathrm{Rev}_n^{\mathrm{Add}} \leq \mathrm{VCG}_{n+C}^{\mathrm{Add}} \leq \mathrm{VCG}_{n+C+m-1}^{\mathrm{DC}}$$

Larger outcome space

Competition complexity $\leq C$

The competition complexity of n additive bidders s.t. identical downward closed constraints over m items is $\leq C + m - 1$.

Extension to downward closed constraints

$$\mathrm{Rev}_n^{\mathrm{DC}} \leq \mathrm{Rev}_n^{\mathrm{Add}} \leq \mathrm{VCG}_{n+C}^{\mathrm{Add}} \leq \mathrm{VCG}_{n+C+m-1}^{\mathrm{DC}}$$

Larger outcome space

Competition complexity $\leq C$

Proved!

The competition complexity of n additive bidders s.t. identical downward closed constraints over m items is $\leq C + m - 1$.

A note on tractability

VCG is not computationally tractable for general downward closed constraints. However:

- VCG is tractable for matroid constraints
- Competition complexity is meaningful in its own right
- Can apply our techniques with “maximal-in-range VCG” by restricting outcomes to matchings

Further extensions (preliminary)

1. From competition complexity to approximation
   - In large markets ($n \gg m$), 2nd price auction (no extra agents) $\frac{1}{2}$-approximates OPT
2. Non-i.i.d. bidders

Summary

- Major open problem: Revenue maximization for m items
- B-K approach: Add competing bidders and maximize welfare
- Results in: First robust simple mechanisms with provably high revenue for many complex settings
- Techniques: Bayesian analysis, combinatorial arguments

Open questions

- Tighter bounds and tradeoffs
  - Settings with constant competition complexity
  - Partial data on distributions, or large markets
  - Different duality based upper bound?
- More general settings
  - Beyond downward closed constraints
  - Irregular distributions
  - Affiliation [Bulow-Klemperer'96]
- Beyond VCG
  - Posted-price mechanisms